Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

Problem 1.

(a) Define what it means for a function defined on $(0,1)$ *to be continuous and to be uniformly continuous on* $(0,1)$.

(b) Prove that if $\{x_n\}$ *is a Cauchy sequence of points in* $(0,1)$ *and* f *is uniformly continuous on* $(0,1)$ *then* $\{f(x_n)\}$ *is a Cauchy sequence. Make sure to include the definition of Cauchy sequence.*

(c) Give an example that shows that the result in part b) is false if we only assume that f *is continuous and explain briefly.*

(d) Define what it means for a function defined on $(0,1)$ *to be absolutely continuous on* $(0,1)$ *(any equivalent definition is acceptable).*

(e) Give an example of a function that is uniformly continuous on $(0,1)$ *but not absolutely continuous on* $(0,1)$*, and explain briefly.*

Problem 2.

(a) Explain from basic definitions how we know that a compact set in $\mathbb{R}^n$ *must be closed and bounded (note that this is the easier direction of the Heine-Borel Theorem). Your answer should include a definition of compact and a definition of closed.*

(b) Suppose that K *is a compact subset of* $\mathbb{R}^n$, $f : K \to K$ *is continuous on* K *and* $\{x_n\}$ *is a sequence of points in* K *with the property that* $|f(x_n) - x_n| < \frac{1}{n}$. *Show that there exists a point* $x \in K$ *such that* $f(x) = x$. *Make it clear what results about compact sets you are using in your argument.*

(c) Is the continuous image of every closed set closed (i.e. is it true that if $f : \mathbb{R}^n \to \mathbb{R}^n$ *and* $F \subset \mathbb{R}^n$ *is closed then* $f(F)$ *is closed)? Explain briefly.*

(d) Is the continuous image of every compact set compact? Explain briefly.

Problem 3.

(a) Outline briefly the construction of Lebesgue measure on the real line. For example we might want to first define it for open and closed bounded sets.

(b) Outline the construction of a set which has measure 1 on $[0,1]$ *but is meager on* $[0,1]$. *Include a definition of meager.*

(c) Give an example to show that it is not true that if $A \subset [0,1]$ *is measurable. then for any* $\epsilon > 0$ *there is a finite collection of intervals* $I_1, I_2, \ldots, I_n$ *such that* $A \subset \cup_{i=1}^n I_i$ *and*

$$\lambda\left((\cup_{i=1}^n I_i) \setminus A\right) < \epsilon.$$

(d) Despite your example to part (c) it is true that every measurable set "can be approximated" by a finite union of intervals in some sense. State a theorem about this and outline a proof.

Problem 4.

(a) *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *be bounded, measurable and positive, and let* E *be a bounded measurable subset of* $\mathbb{R}^n$. *Outline how we define the Lebesgue integral* $\int_E f$. *Make it clear where we are using the fact that* f *is a measurable function.*

(b) *Give an example of a sequence of measurable functions* $f_n : [0,1] \to \mathbb{R}$ *such that the* f_n *converge to a function* f *pointwise on* $[0,1]$ *but* $\int_{[0,1]} f_n$ *does not converge to* $\int_{[0,1]} f$.

(c) *Discuss and explain the dominated convergence theorem. Your example in part b) must not satisfy all the conditions (or else the integrals would have converged correctly). Discuss this briefly.*

(d) *Illustrate that the dominated convergence theorem does not work for Riemann integration by giving an example in which all the conditions are satisfied and yet* $\int_{[0,1]} f$ *does not exist.*

Problem 5.

(a) *Give a concise construction of the field of complex numbers* $\mathbb{C}$ *(in particular, your answer should explain why the resulting algebraic structure is indeed a field).*

(b) *Explain the operations of addition and multiplication of complex numbers from a geometric point of view.*

(c) *How do you extend the usual real function* $f(x) = \sin(x)$ *to a complex function* $f(z) = \sin(z)$? *How do you know that the resulting function is* differentiable?

(d) *How would you compute* $\sin(\mathrm{i})$?

(e) *Find all* z *such that* $\sin(z) = \mathrm{i}$. *Do it in two different ways: using a purely real-variable approach (work separately with the real and the imaginary parts), and then using a complex variable approach (i.e., inverse functions). Make sure to get the same answer.*

Problem 6.

(a) *Give a definition of what it means for a function* $f(z)$ *of a complex variable to be* differentiable *at a point* z_0. *Explain why* differentiability *is equivalent to* linearizability *of* $f(z)$.

(b) *Explain what the Cauchy-Riemann equations are.*

(c) *Find the derivative of a function* $f(z) = z^2 + 6z$ *at a point* z_0 *directly from the definition and by using the Cauchy-Riemann equations, then match your answers.*

(d) *Describe geometrically the local structure of the map* $f(z) = z^2 + 6z$ *at the point* $z_0 = 3\mathrm{i}$.

(e) *Where is the map* $f(z) = z^2 + 6z$ *not conformal? What is the geometric structure of* $f(z)$ *at those points?*

Problem 7. *Consider the function* $f(z) = \dfrac{2z+3}{z^2+4}$ *and the mapping* $w = f(z)$ *given by this function (extended to the Riemann sphere* $\mathbb{CP}^1$*).*

(a) *What is the degree of this map? How many preimages would a typical point* $w \in \mathbb{CP}^1$ *have?*

(b) *Find all images of* $z = \infty$. *Find all preimages of* $w = \infty$.

(c) *What are the points* w *that have fewer preimages than generic points? What is happening there?*

Problem 8.

(a) *State* Cauchy's Integral Theorem *and* Cauchy's Integral Formula. *Give a sketch of a proof for both (this can be informal, just try to explain the main ideas).*

(b) *Use* Cauchy's Integral Formula *to evaluate*

$$\oint_C \frac{z-1}{(z^2+1)(z+1)^2}\, dz,$$

where C is a circle centered at $1+\mathrm{i}$ and passing through the origin (oriented counter-clockwise).

Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

Problem 1.

(a) *State (without proof) three equivalent ways to describe compactness of subsets of the Euclidean plane.*

(b) *Prove that two of these descriptions are equivalent. (You may choose which two.)*

(c) *Prove that the Cartesian product $K_1 \times K_2$ of two compact subsets $K_1, K_2 \subset \mathbb{R}$ is a compact subset of the plane.*

Problem 2. *Let D be a subset of the Euclidean plane..*

(a) *Explain what it means for D to be* connected; arcwise connected.

(b) *Prove that if D is* arcwise connected, *then it is* connected.

(c) *Prove that if D is open, then the converse statement is also true.*

(d) *Give an example of D that is connected but not arcwise connected.*

Problem 3.

(a) *Define the following concepts, paying especial attention to the dependence/independence of your quantified variables:*

- *uniform continuity of a function f;*
- *uniform convergence of a sequence of functions $f_n \xrightarrow{\text{unif}} f$.*

(b) *Prove, or disprove by giving a counter-example: if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both uniformly continuous maps, then the composite function $f \circ g$ is uniformly continuous.*

(c) *Prove, or disprove by giving a counter-example: if $f_n : \mathbb{R} \to \mathbb{R}$ and $g_n : \mathbb{R} \to \mathbb{R}$ are functions that converge uniformly to limit functions f and g respectively as $n \to \infty$, and if f is uniformly continuous, then the composite functions $f_n \circ g_n$ converge uniformly to $f \circ g$ as $n \to \infty$. (Do not assume that f_n or g_n are necessarily continuous.)*

Problem 4. *State and prove the* Intermediate Value Theorem *for a real-valued function f whose domain is a subset of the real line.*

Problem 5.

(a) *List the axioms that define a σ-algebra.*

(b) *Define the Borel σ-algebra.*

(c) *Prove that every Lebesgue-measurable subset M of the real line is contained in a Borel set B such that $B \setminus M$ has measure zero. (Make sure to define what it means for a subset of the real line to have Lebesgue measure zero).*

Problem 6. *Outline the steps in the construction of a fat Cantor set: a closed subset $K \subset [0, 1]$ with no interior points that has positive Lebesgue measure.*

Problem 7.

(a) *Give a concise construction of the field of complex numbers $\mathbb{C}$ (in particular, your answer should explain why the resulting algebraic structure is indeed a field).*

(b) *Explain, in some detail, why the multiplication by a complex number z with $|z| = 1$ corresponds geometrically to a rotation of the complex plane.*

(c) *Explain the meaning of 2^{i}. In particular, how many points on the complex plane does this expression describe? What about i^2?*

(d) *Solve the equation $z^3 = \mathrm{i}$ using both the complex analysis approach and the real variables approach (and make sure to explain why you get the same answers).*

Problem 8.

(a) *Give a definition of what it means for a function $f(z)$ of a complex variable to be differentiable at a point z_0.*

(b) *Explain what the Cauchy-Riemann equations are.*

(c) *Carefully prove that the Cauchy-Riemann equations give a necessary and sufficient condition for the differentiability of $f(z)$.*

(d) *Find the derivative of a function $f(z) = z^3$ at a point $z_0 \neq 0$ directly from the definition and by using the Cauchy-Riemann equations, then match your answers.*

Problem 9. *Consider the function $f(z) = z^2 - 2z$ and the mapping $w = f(z)$ given by this function (extended to the Riemann sphere).*

(a) *Describe all points at which the mapping given by $f(z)$ is* conformal.

(b) *What happens at the points where $f(z)$ is not conformal?*

(c) *Which part of the complex plane is* shrunk *and which part is* stretched *under this mapping?*

(d) *Find the angle of rotation and the local magnification factor of this mapping at the point $z = \mathrm{i}$.*

(e) *How many pre-images would a typical point on the w-plane have (give an example)? Which points are exceptional in that respect?*

Problem 10.

(a) *Prove that, if f is continuous in a neighborhood of the origin $z = 0$,* $\lim_{r\to 0} \int_0^{2\pi} f\left(re^{\mathrm{i}\theta}\right)\, d\theta = 2\pi f(0)$.

(b) *Let $C_r(a)$ denote the positively-oriented circle of radius r centered at a. Using the* Cauchy Integral Formula, *evaluate the following integrals.*

$$\oint_{C_3(\mathrm{i})} \frac{1}{z^2+9}\, dz \qquad \text{and} \qquad \oint_{C_3(\mathrm{i})} \frac{1}{(z^2+9)^2}\, dz$$

Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

Problem 1. *State two or more equivalent characterizations of compactness of a set $K \subset \mathbb{R}$ and prove that two such characterizations are indeed equivalent.*

Problem 2. *For each of the following, either give a specific example, or tell why no example can exist. (Assume that f is a real-valued function.)*

(a) A function f that is continuous, but not uniformly continuous on the interval $[0,1]$.

(b) A function f that is continuous on the interval $[0,1]$, and $f(r) \leq 1$ for all rational numbers $r \in [0,1]$, but $\sup_{x\in[0,1]} f(x) > 1$.

*(c) A continuous function f mapping $(-1,1)$ **onto** $(-1,0) \cup (0,1)$. (*Remark*: A function f maps a set E* onto *a set F provided that $f(E) = F$.)*

True or False (Justify your claim or provide a counterexample)

(d) If $f : X \to Y$ is a continuous function from a metric space X to a metric space Y and $K \subseteq Y$ is compact, then $f^{-1}(K) \subseteq X$ is compact.

(e) If f is a continuous real-valued function on $X \subseteq \mathbb{R}$ and (x_n) is a sequence in X such that $f(x_n)$ converges, then (x_n) converge.

Problem 3. *Let $C \subset [0,1]$ denote the standard middle-thirds Cantor set.*

(a) Provide a short description of the construction of C.

(b) Is C closed or not closed? Justify your answer.

(c) Is C countable or uncountable? Justify your answer.

(d) Does C contain any intervals of positive length? Justify your answer.

(e) Is the complement of C a dense subset of $[0,1]$ or not dense? Justify your answer.

Problem 4. *State the axioms that define an abstract measure space $(X, \mathfrak{M}, \mu)$ where $\mathfrak{M} \subset 2^X$. Be sure to give a detailed description of the axiomatic properties that must be satisfied by the $\sigma-$algebra denoted as $\mathfrak{M}$.*

Problem 5.

(a) What is the definition of a Lebesgue measurable function $f : \mathbb{R} \to [0, \infty]$?

(b) Prove that the point-wise limit of every monotone increasing sequence of such measurable functions is another measurable function; that is, if each function

$$f_1(x) \leq f_2(x) \leq \ldots \leq f_n(x) \leq f_{n+1}(x) \leq \ldots$$

is measurable and takes values in $[0, \infty]$, then $f(x) = \lim_{n\to\infty} f_n(x)$ is also measurable.

Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

Problem 1. *Give a definition of a* connected *set (if you prefer, you can assume that your sets are subset of the complex plane). Prove that if A and B are connected and $A \cap B \neq \varnothing$, then $A \cup B$ is connected. What about $A \cap B$?*

Problem 2.

(a) Give a definition of what it means for a function $f(z)$ of a complex variable to be differentiable at a point z_0.

(b) Explain what the Cauchy-Riemann equations are.

(c) Carefully prove that the Cauchy-Riemann equations give a necessary and sufficient condition for the differentiability of $f(z)$.

(d) Find the derivative of a function $f(z) = z^{-1}$ at a point $z_0 \neq 0$ directly from the definition and by using the Cauchy-Riemann equations, then match your answers.

(e) Write down the Cauchy-Riemann equations using the polar coordinates in the domain of f.

(f) Can you use the polar form of the Cauchy-Riemann equations to find the derivative of the distance function $f(z) = |z^2| = r^2$?

Problem 3.

(a) Discuss the phenomena of multi-valuedness for functions of a complex variable. Why is it necessary? How do we work with it? How is this situation different from the real-variable case?

(b) Find all values of the following:

- $(1 - \mathrm{i})^{1+\mathrm{i}}$;
- $2^{\sqrt{3}}$;
- $\mathrm{i}^{2/3}$ *(how many distinct values are there? Can you sketch them on the complex plane?)*

Problem 4.

(a) Explain the notion of a conformal map.

(b) Consider the function $f(z) = z^2$ on the Riemann Sphere $\mathbb{CP}^1$. Show that this function defines a conformal map at the point $z = \mathrm{i}$. Describe the local structure of this map.

(c) What are the points where $f(z)$ fails to be conformal? What happens at those point?

Problem 5.

(a) Explain what are Linear Fractional Transformations *and show that they form a group.*

(b) Explicitly describe the subgroup of this group consisting of LFTs *that permute the points 0, i, and ∞.*

(c) Explicitly verify that composition and inverse operations for some elements of this group (one non-trivial example of each).

Problem 6. *Let $f(z)$ be a continuous (but not necessarily analytic) function defined in the neighborhood of $z = 0$ and let C_r be a circle of radius r around the origin in the complex plane. Carefully prove that*

$$\frac{1}{2\pi} \lim_{r\to 0} \oint_{C_r} f(z)\,dz = f(0).$$

Problem 7.

(a) Find all roots of the polynomial $p(z) = z^4 + 4z^2 + 1$ and sketch their locations in the complex plane.

(b) Evaluate

$$\int_0^\infty \frac{dx}{x^4 + 4x^2 + 1}$$

using the Cauchy Integral Theorem. *Briefly, but carefully, justify the main steps in your reasoning.*