Comprehensive Examination in Analysis
Ph.D. in Educational Mathematics
University of Northern Colorado
Summer 2018

Please answer as many of the following questions as you can in the time allotted. Although you may not be able to answer all the questions, passing will require you to show breadth of knowledge (answer a variety of questions), depth of understanding (answer some questions that require explanation), and ability to write correct proofs.
The UNC honor code applies to this exam. You may not consult any books, notes, online resources, computing devices, or other aids during this exam, nor collaborate with any other person.

1. Here is one version of the intermediate value theorem:

   Theorem. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and suppose $a < c$ are two real numbers such that $f(a) < f(c)$. For every number $y \in (f(a), f(c))$, there exists a number $b \in (a, c)$ with $f(b) = y$.

   (a) Sketch a proof of this theorem.

   (b) Explain why the continuity of f is essential for this theorem, and point out where it is used in your proof.

   (c) Explain why the completeness of $\mathbb{R}$ is essential for this theorem, and point out where it is used in your proof.

2. For each part, either:

   - Give an example of a subset of $\mathbb{R}$ that has all of the given properties, and briefly explain how you know that it does; or
   - Prove that no such set exists.

   (a) Dense and meager

   (b) Not dense and not meager

   (c) Meager, and its complement is countable

   (d) Meager and positive measure

   (e) Measure zero and uncountable

3. With justification, determine the value of the two-dimensional Lebesgue integral $\int_R f\, dm$ where R is the rectangle $[0, 1] \times [0, 2\pi]$ and $f(x, y) = \sin(x^2 + y)$.

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4. (a) State Fatou's lemma, and sketch how it can be proved using the monotone convergence theorem.
   (b) Does there exist a sequence of functions $f_n : \mathbb{R} \to [0, \infty)$ converging pointwise to a function f, such that $\int f_n \, dm = 1$ for all n, but $\int f \, dm = 0$? Either find an example of such a sequence, or prove that none exists.
   (c) Does there exist a sequence of functions $f_n : \mathbb{R} \to [0, \infty)$ converging pointwise to a function f, such that $\int f_n \, dm = 1$ for all n, but $\int f \, dm = 2$? Either find an example of such a sequence, or prove that none exists.
   (d) Does there exist a sequence of functions $f_n : [0, 1] \to [0, 5]$ converging pointwise to a function f, such that $\int f_n \, dm = 1$ for all n, but $\int f \, dm = 0$? Either find an example of such a sequence, or prove that none exists.

5. Let $V \subset [0, 1]$ be the non-measurable Vitali set.
   (a) Briefly describe how V is constructed.
   (b) What key properties of V are used in showing that it cannot be measurable?
   (c) Suppose A is a measurable set contained in V. Prove that A has measure zero.

6. For what values of z is the complex-valued function $w = f(z) = \dfrac{\overline{z}^2}{2} - \dfrac{\overline{z}z^3}{3}$
   (a) continuous?
   (b) real differentiable?
   (c) complex-differentiable?

   Justify your answers.

7. Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{2^n n!}{\sqrt{(2n)!}} z^n.$$

8. Let $f(z) = \tan\left(\dfrac{\pi}{z^2}\right)$.
   (a) Locate all isolated singularities of f.
   (b) Choose one of the isolated singularities of f that is on the imaginary axis, and find the residue of f at that point.

9. Complete **ONE** of the following three possible short essay problems. Submit only one solution, with a suggested length of no more than one page.
   (a) State Liouville's Theorem and sketch a proof.
   (b) State a version of the Fundamental Theorem of Algebra that is valid for all analytic polynomials, and sketch a complex-analytic proof of that version.
   (c) State a version of Cauchy's Theorem and give an outline of its proof.

End of the exam