Algebra Comprehensive exam

August 5, 2014

Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show breadth of knowledge (answer a variety of questions), depth of understanding (answer some questions that require explanation) and ability to write correct proofs.

1. Define the following terms, and illustrate each one with an appropriate non-trivial example:
   (a) left coset
   (b) normal subgroup
   (c) group homomorphism
2. Which elements of Z_n are cyclic generators of the whole group? Prove your answer.
3. Prove that if the order of a group is a power of a prime, then the group has a non-trivial center.
4. (a) Give an example of an order $n > 25$ such that there is exactly one group of order n up to isomorphism. Prove your answer.
   (b) Give an example of an order $n > 25$ such that there are exactly two groups of order n up to isomorphism. Prove your answer.
   (c) Give an example of an order $n > 25$ such that there are at least four groups of order n up to isomorphism. Prove your answer.
5. Use the Sylow theorems to show that, up to isomorphism, there is only one group of order 15.

6. Let G be an abelian group of order n, and let p be a prime factor of n. Use strong induction on n to show that G contains an element of order p.

7. Show from basic principles that multiplication is not well-defined on the cosets of $\mathbb{Z}[x]$ in $\mathbb{Q}[x]$.

8. (a) Give an example of an integral domain that is not a unique factorization domain, and explain briefly.

   (b) Outline a proof that any $F[x]$ is a unique factorization domain (where F is any field).

9. (a) Explain the fundamental homomorphism theorem for rings. In your explanation, refer to a specific homomorphism from $\mathbb{Q}[x]$ into $\mathbb{R}$ that has kernel equal to $(x^5 - 2)$.

   (b) Assuming ϕ is the homomorphism from part a), what specific real number is $\phi(x^7 - x^5 + 3)$? Explain briefly.

   (c) Explain carefully how we know that $\mathbb{Q}[x] \;/(x^5 - 2)$ is a field. Your explanation should involve properties of polynomials, and not simply quote a major result.

   (d) Use parts a) and c) to explain how we know that $\mathbb{Q}\left(\sqrt[5]{2}\right) = \left\{a_0 + a_1\sqrt[5]{2} + a_2\sqrt[5]{2}^2 + a_3\sqrt[5]{2}^3 + a_4\sqrt[5]{2}^4 : a_0, ..., a_4 \in \mathbb{Q}\right\}$ is a field.

10. (a) Define the term "*Galois Group*" of a field K over a field F.

    (b) Give an example to show that this notion would not be interesting for every extension field K of F (thus showing why you needed a condition on the type of extension field in part a). Clarify why your example works.

    (c) Describe the Galois group of the splitting field of $(x^3 - 2)(x^2 - 5)$ in detail. This should include a known group theoretic way to write the group and some indication of what the elements look like in terms of permuatations of the roots.

    (d) Find a non-normal subgroup of this Galois group and a non-trivial normal subgroup and their corresponding fixed fields.

    (e) Use your examples in d) to explain the Fundamental Theorem of Galois Theory.

Algebra Comprehensive Exam
August 9, 2013

Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show breadth of knowledge (answer a variety of questions), depth of understanding (answer some questions that require explanation) and ability to write correct proofs.

1. Define the following terms:

   (a) subgroup

   (b) coset

   (c) p-Sylow subgroup

2. (a) Explain why the cosets of a group may not always form a group themselves under the natural operation inherited from the group, using an appropriate example.

   (b) Define a condition under which the cosets do form a group, and prove that they do always form a group under this condition. Illustrate this with an appropriate example.

3. Prove that the only subgroup of D_5 that contains two reflections is D_5 itself.

4. (a) Define the **center** of a group.

   (b) Define the **normalizer** of a in G.

   (c) State and explain the class equation, using any non-abelian group to illustrate.

   (d) Use the class equation to prove that if G is a group of order 175 that has a trivial center, then G contains an element that commutes with exactly 25 other elements.

5. Find all integers n such that there exists an onto homomorphism from D_7 onto Z_n. Prove that your answer is correct.

6. Give examples of the following and show why they are examples OR explain why no such example exists.

   (a) A polynomial of degree 6 that is irreducible in $\mathbb{Z}$.

   (b) A polynomial of degree 6 that is irreducible in $\mathbb{R}$.

   (c) An integral domain that is not a unique factorization domain.

   (d) A subring that is not an ideal in $\mathbb{Z}[x]$.

   (e) A field extension of $\mathbb{Q}$ of degree 5 that is not the splitting field of any polynomial over $\mathbb{Q}$.

7. Let r be a root of the polynomial $p(x) = x^3 - 2x^2 + 3x + 1$.

   (a) Prove that $\mathbb{Q}[x]/(p(x)$ is a field from basic polynomial principles. In the process of doing this you should clarify what multiplication looks like in this setting with an example, and focus on showing clearly why every element must have an inverse. The other field properties should be identified but little needs to be written about most of them.

   (b) Use part a) to prove that $\mathbb{Q}(r)$ is a field.

8. (a) Define the *Galois Group* of a field K over a field F.

   (b) Explain why the definition in part a) does not apply (or is not interesting) if $K = \mathbb{Q}(\sqrt[3]{2})$ and $F = \mathbb{Q}$.

   (c) Suppose that K is degree 75 over F and is the splitting field of some polynomial over F. Explain using Galois theory and group theory why there must be a subfield E of K that is degree 3 over F and is the splitting field of some polynomial over F.

   (d) Describe the Galois group of the splitting field of $(x^4 - 2)(x^2 - 3)$ in detail. This should include a known group theoretic way to write the group and some indication of what the elements look like in terms of permutations of the roots, although you do not need to write down all elements.

   (e) Find a normal subgroup of order 4 of the group from part d) and find its fixed field. Explain how we know that this answer is correct.

Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

Problem 1.

(a) Give the definition of a group.

(b) Let $\mathbf{V}$ *and* $\mathbf{W}$ *be two vector spaces over some field* $\mathbb{K}$.

- *Does the set of all* linear *maps* $L : \mathbf{V} \to \mathbf{W}$ *form a group w.r.t. the operation of* composition*?*
- *If not (and then explain why not), what conditions should be imposed so that we get an example of a (nontrivial) group?*

(c) Let $\mathcal{F} = \{F : \mathbb{R} \to \mathbb{R}\}$ *be the set of all* smooth *real-valued functions of a real variable. Is this set a group w.r.t. operations of*

- *composition?*
- *multiplication?*
- *addition?*

In each case, carefully explain why *or* why not.

Problem 2. *Let* G *be a group and* H *a subset of* G.

(a) Explain what it means for H *to be a* subgroup *of* G.

(b) Define a coset *of* H *in* G.

(c) Prove that cosets partition *the group* G *(and make sure to explain what it means). How is this statement related to* Lagrange's Theorem*?*

(d) Explain why, for any set S, *a* partition *of* S *is equivalent to defining an* equivalence relation *on* S *(make sure to explain what an equivalence relation is).*

(e) What is the equivalence relation that corresponds to the partition of G *by right cosets of* H*? Carefully prove that it is indeed an equivalence relation.*

(f) Is it true or false that any partition of G *corresponds to a partition by cosets of some subgroup* H *of* G*?*

Problem 3.

(a) Define what it means for two subgroups H_1 *and* H_2 *of a group* G *to be* conjugated *in* G.

(b) Show that if H_1 *and* H_2 *are conjugated in* G, *then they are necessarily isomorphic (and make sure to explain what it means). Is the converse true?*

(c) Consider the dihedral group D_4 *of symmetries of a square (this group is sometimes also denoted by* D_8*). Show that the subgroup* H_1 *generated by the reflection about the main diagonal and the subgroup* H_2 *generated by the reflection about the horizontal axes of symmetry are* not conjugated *in* D_4, *but* H_2 *and the subgroup* H_3 *generated by the reflection about the vertical axes of symmetry are* conjugated *in* D_4.

(d) Consider the dihedral group D_3 *of symmetries of a equilateral triangle. Show that all three subgroups* H_i *generated by reflections* m_i *about the symmetry axes of the triangle are conjugated in* D_3.

(e) Interpret your conclusions in parts (c) and (d) using (some) Sylow Theorems *(and make sure to carefully state the necessary theorems and the accompanying definitions).*

Problem 4.

(a) *Define an* automorphism *of a group* G*.*

(b) *Explain, briefly, why the set* $\mathrm{Aut}(G)$ *of all automorphisms of* G *is a group w.r.t. the operation of composition.*

(c) *What is an* inner automorphism *of* G*?*

(d) *For* $G \simeq \mathbb{Z}_8$*, which familiar groups are* $\mathrm{Aut}(\mathbb{Z}_8)$ *and* $\mathrm{Inn}(\mathbb{Z}_8)$ *isomorphic to?*

Problem 5.

(a) *Define a normal subgroup of a group.*

(b) *Show that there is a one-to-one correspondence between normal subgroups of a group* G *and kernels of surjective group homomorphisms from* G*.*

(c) *Give an example of a group all of whose subgroups are normal.*

(d) *Prove that any subgroup* $H < G$ *of index* 2 *is normal.*

(e) *What is an example of a (non-trivial) normal subgroup* K *of the permutation group* $\mathfrak{S}_5$*? What is* $\mathfrak{S}_5/K$*?*

Problem 6.

(a) *Without explicitly listing the elements, determine the cardinality of the group* T *of rotational symmetries of the regular tetrahedron.* ***Hint****: use the orbit/stabilizer count.*

(b) *Describe the elements of* T*.*

(c) *In how many essentially different ways can we color the faces of a regular tetrahedron in three different colors?* ***Hint****: use Burnside's theorem.*

Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

Problem 1. *Give examples of the following and show why they are examples OR explain why no such example exists.*

(a) An integral domain with exactly 4 units (invertible elements).

(b) A ring that is not an integral domain with exactly 4 units.

(c) A polynomial of degree 3 that is irreducible in $\mathbb{Z}_3[x]$

(d) A polynomial of degree 3 that is irreducible over some field (i.e. in some $F[x]$*) but not over a larger field (specify the polynomial and the two fields).*

(e) A polynomial of degree 3 that is irreducible over $\mathbb{Q}[x]$ *but not over* $\mathbb{Z}[x]$.

(f) A subring that is not an ideal in $\mathbb{Q}[x]$.

(g) A subring that is not an ideal in $\mathbb{Z}$

Problem 2.

(a) Give a definition of a ring homomorphism *and the* kernel *of a ring homomorphism.*

(b) Give a definition of an ideal *and a* maximal ideal *in a ring.*

(c) Let R *be a commutative ring and let* I *be an ideal in* R. *Explain from basic principles and with an example why multiplication is well-defined in* R/I.

(d) Let D *be a an integral domain and* I *be an ideal in* D. *Show from basic principles (i.e. don't use a big theorem that makes this easy) that every non-zero element in* D/I *has an inverse iff* I *is a maximal ideal.*

(e) Explicitly show the multiplication for $\mathbb{Q}[x]/(x^2 - 5x + 6)$, *i.e. letting* $I = (x^2 - 5x + 6)$, *show what* $(ax + b + I)(cx + d + I)$ *is equal to in the quotient ring.*

(f) Are there any zero divisors in the quotient ring $\mathbb{Q}[x]/(x^2 - 5x + 6)$*? Explain.*

Problem 3.

(a) Outline the proof that the set $\mathbb{Q}(\sqrt[3]{2}) = \{a + b(\sqrt[3]{2}) + c(\sqrt[3]{2})^2 : a, b, c \in \mathbb{Q}\}$ *is a field. Your proof should include the defining of a homomorphism* ϕ *from* $\mathbb{Q}[x]$ *onto* $\mathbb{Q}(\sqrt[3]{2})$.

(b) What specific element in $\mathbb{Q}(\sqrt[3]{2})$ *is* $\phi(x^4 - 2x^3 + x^2 + 4x + 1)$, *assuming* ϕ *is the same homomorphism you defined when you answered part a) above?*

(c) Define the degree *of a field extension (using the concept of dimension) and state and explain what the degree of the extension field* $\mathbb{Q}(\sqrt[3]{2})$ *over* $\mathbb{Q}$ *is.*

(d) Is $\mathbb{Q}(\sqrt[3]{2})$ *a splitting field of any polynomial over* $\mathbb{Q}$*? Explain why or why not. If not, say what element could be adjoined to make it a splitting field.*

(e) The complex number c *is not equal to* $\sqrt[3]{2}$ *but* $\mathbb{Q}(\sqrt[3]{2})$ *is isomorphic to* $\mathbb{Q}(c)$. *Find all possible values of* c *and explain.*

Problem 4.

(a) *Find or explain how to construct a field of order 6 or explain why none exists.*

(b) *Find or explain how to construct a field of order 7 or explain why none exists.*

(c) *Find or explain how to construct a field of order 8 or explain why none exists.*

Problem 5.

(a) *Give a definition of the* Galois group *of a field* $\mathbb{K}$ *over a field* $\mathbb{F}$.

(b) *Let* G *be the Galois group of the splitting field of* $x^4 - 3$ *over* $\mathbb{Q}$. *Identify this group in more familiar group notation of your choice.*

(c) *Find the subgroup of* G *in your notation that corresponds to the field* $\mathbb{Q}(\sqrt{3}i)$ *via the correlation given by the Fundamental Theorem of Galois Theory, and explain.*

(d) *Find the field extension in the lattice that is not a splitting field over* $\mathbb{Q}$, *correlate it with the appropriate subgroup, and explain the concept from the fundamental theorem that this illustrates.*

Summer 2010 Algebra Comprehensive Examination, Part I

Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

1)Define the following terms, and give an appropriate example (you do not need to prove that your examples are correct).

a) A *kernel* of a homomorphism from one group to another. Give an example of a homomorphism from $\mathbf{D}_4$ to some other group that has a kernel of order 4.

b) An *automorphism* of a group. Give an example of a nontrivial (not the identity map) automorphism of S_3.

c) What it means for two subgroups to be *conjugate*. Give an example of two (unequal) conjugate subgroups of S_4.

d) The *normalizer* or *centralizer* of a group. Let g be the element of D_4 that corresponds to reflection about the y axis. What is the normalizer of this element?

2)a) Define what it means for two groups to be isomorphic.

b) Prove that if G_1 is isomorphic to G_2, and G_1 has an element of order 4 then G_2 has an element of order 4.

c) Is the result in part b) still true if we replace "isomorphic to" with "is the homomorphic image of?" If it is true, prove it. If it is false, give an example.

3)a) State the three Sylow theorems.

b)Use the Sylow theorems to show that if G is a group of order 30 then it has either a normal subgroup of order 3 or a normal subgroup of order 5.

c) Suppose that G is a group of order 24. Use the Sylow theorems to describe the possible number of subgroups of order 3 that G might have. Find examples that show that these possibilites actually occur.

4)a) State the first isomorphism theorem.

b) Explain the theorem in detail using a homomorphism from A_4 onto $\mathbf{Z}_3$ as an example.

c) Use the first isomorphism theorem and other group theory results to show that if G is a group of order 100 that has a homomorphism onto $\mathbf{Z}_5 \times \mathbf{Z}_5$ then G is abelian.

5)a) Explain the class equation, using $S_3 \times \mathbf{Z}_2$ as an example.

b) Use the class equation to prove that every group of order 81 has a non-trivial center.

Summer 2010 Algebra Comprehensive Examination, Part II

Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

1) a) Define *Ring.*

b) Define *Integral Domain.*

c) Define *Field.*

d) Discuss the similarities and difference between these three mathematical objects.

2) Give examples of the following (and show why they are examples):

a) An integral domain that is not a unique factorization domain.

b) An ideal in a ring that is maximal but not prime (make it clear what ring you are using for this example).

c) An ideal that is not principal in some ring (make it clear what ring you are using for this example).

d) A field F with the property that there exists only one automorphism of F

3) a) Define *ring homomorphism* and the *kernel* of a ring homomorphism.

b) Is every subring of the integers the kernel of a ring homomorphism? Explain.

c) Is every subring of $\mathbf{Q}[x]$ the kernel of a ring homomorphism? Explain.

d) Discuss the relationship in general between subrings and kernels of ring homomorphisms.

4) Let ϕ be the homomorphism that sends $\mathbf{Q}[x]$ into the real numbers $\mathbf{R}$ by evaluation at $\sqrt[3]{5}$. So $\phi(a_0 + a_1 x + a_2 x^2 + ... + a_n x^n) = a_0 + a_1(\sqrt[3]{5}) + a_2(\sqrt[3]{5})^2 + ... + a_n(\sqrt[3]{5})^n$.

a) Is this homomorphism surjective (onto)?

b) Describe the kernel K of this homomorphism and prove that your answer is correct.

c) EXPLICITLY describe the quotient ring $\mathbf{Q}[x]/K$. Your answer should be in terms of the way that quotient rings are constructed and not dependent on more advanced theorems.

d) What equivalence class of the form $[a + bx + cx^2]$ is $[2 + 3x + 5x^2 + 7x^3]$ in in this ring?

e) Relate your answer to c) to the answer we obtain from the fundamental isomorphism (homomorphism) theorem for rings.

5 a)Define the *splitting field* of a polynomial $p(x)$ over a field F.

b) Define the *degree* of a field extension.

c) Find the degrees of each of the following three fields over $\mathbf{Q}$:
$\mathbf{Q}[\sqrt{5}+i]$, $\mathbf{Q}[\sqrt{5}\cdot i]$, $\mathbf{Q}[\sqrt{5}, i]$

d) Are any of the three fields in part c) above equal to each other (i.e. literally the same field)? Explain.

6) a) Define the *Galois Group* of a field K over a field F.

b) Suppose that $p(x)$ is a polynomial of degree 5 in $\mathbf{Q}[x]$ whose splitting field E is degree 120 over $\mathbf{Q}$. Use the Fundamental Theorem of Galois theory and group theory results to show that there is a subfield K of E that is degree 15 over $\mathbf{Q}$.

c) Is the field K from part b) also a splitting field of some polynomial over $\mathbf{Q}$? Explain.

d) With G and E as in part b) explain how we know that there any subfield of E of that is itself a splitting field of some polynomial over $\mathbf{Q}$ must be of degree 2 over $\mathbf{Q}$.

Summer 2009 Algebra Comprehensive Examination, Part I

Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

1)Define the following terms, and give an appropriate example:

a) A *homomorphism* from one group to another. Give an example of a homomorphism from $\mathbf{S}_3$ to some $\mathbf{Z}_n$, where n is not 1.

b) What it means for two subgroups to be *conjugate*. Give an example of two different subgroups of D_4 that are conjugate to each other.

c) A *Sylow p-subgroup*. Give an example of a Sylow 2-subgroup of A_4.

d) An *automorphism* of a group. Give an example of a non-trivial automorphism of $\mathbf{Z}_6$.

2)a) Define what the *order* of an element of a group means.

b) Prove that if G_1 is isomorphic to G_2, and G_1 has an element of order 3 then so does G_2.

c) Is there an element of order 15 in S_8? Is there an element of order 14? Explain briefly.

3)a) State the three Sylow theorems.

b)Use the Sylow theorems to prove that every group of order 45 is abelian.

c)Use the Sylow theorems to find the number of elements of order 2 in a non-abelian group of order 14. Explain.

4)a) State the first isomorphism theorem.

b) Explain the theorem in detail using a homomorphism from D_4 onto $\mathbf{Z}_2 \times \mathbf{Z}_2$ as an example.

c) Use this theorem and other group theory results to prove carefully that there is no homomorphism from S_4 onto $\mathbf{Z}_4 \times \mathbf{Z}_2$.

5)a) Prove that the set of all automorphisms of a group is itself a group (with composition being the operation). You do not need to prove associativity of the composition operation.

b) Consider the following statement: the set of all automorphisms of a group is isomorphic to the group itself. Is this statement true for all groups? Is it false for all groups? Explain as clearly as possible and with examples.

6) Explain the class equation using $S_3 \times \mathbf{Z}_2$ as an example.

Summer 2009 Algebra Comprehensive Examination, Part II

Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

1) a) Define *group.*

b) Define *Ring.*

c) Define *Field.*

d) Discuss the similarities and difference between these three mathematical objects.

2) Give examples of the following (and show why they are examples):

a)A principal ideal in $\mathbf{Q}[x]$.

b) Is there an example of an ideal in $\mathbf{Q}[x]$ that is not principal? If so, give one. If not, prove that no such ideal exists.

c) An ideal that is not principal in some ring (make it clear what ring you are using for this example).

3) a) Define *ring homomorphism.*

b) Define the *kernel* of a ring homomorphism.

c) Define a *subring* of a ring.

d) Give an example of a subring that is not the kernel of any homomorphism (again, make it clear what ring you are working in).

e) What properties distinguish kernels of homomorphisms from other rings?

f) Is every subring with the additional properties you have listed in part e) a kernel of some homomorphism? Explain.

4) Let ϕ be the homomorphism that sends $\mathbf{Q}[x]$ into the complex numbers $\mathbf{C}$ by evaluation at i. So $\phi(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n) = a_0 + a_1i + a_2i^2 + \ldots + a_ni^n$.

a) Is this homomorphism surjective (onto)?

b) Describe the kernel K of this homomorphism and prove that your answer is correct.

c) EXPLICITLY describe the quotient ring $\mathbf{Q}[x]/K$. Your answer should be in terms of the way that quotient rings are constructed and not dependent on

d) What equivalence class of the form $[a + bx]$ is $[2 + 3x + 5x^2]$ equal to in in this ring?

e) Relate your answer to c) to the answer we obtain from the fundamental isomorphism (homomorphism) theorem for rings.

5 a)Define the *splitting field* of a polynomial $p(x)$ over a field F.

b) Define the *degree* of a field extension over another field.

b) Are the degrees of the splitting fields of the polynomials $x^3 - 1$ and $x^3 - 3$ the same over $\mathbf{Q}$? Explain.

d) Find the degrees of each of the following fields over $\mathbf{Q}$:
$\mathbf{Q}[\sqrt{3} + \sqrt{6}]$, $\mathbf{Q}[\sqrt{3} \cdot \sqrt{6}]$, $\mathbf{Q}[\sqrt{3}, \sqrt{6}]$

e) Are any of the three fields in part d) above equal to each other (i.e. literally the same field)? Explain.

6) a) Define the *Galois Group* of a field K over a field F.

b) Suppose that $p(x)$ is a an polynomial of degree 5 in $\mathbf{Q}[x]$ with the property that its splitting field has degree 40 over $\mathbf{Q}$. Use the Fundamental Theorem of Galois Theory and Sylow's theorems to show that there is a field extension of $\mathbf{Q}$ of degree 4 that is normal over $\mathbf{Q}$ and contained in the splitting field of $p(x)$.

Summer 2008 Algebra Comprehensive Examination, Part I

Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

1)Define the following terms, and give an appropriate example:

a) A *homomorphism* from one group to another. Give an example of a homomorphism from $\mathbf{Z}_{12}$ to some other $\mathbf{Z}_n$, where n is not 1 or 12.

b) A *normal* subgroup. Give an example in which the normal subgroup is neither the whole group nor of order 1, and in which the group itself is not abelian.

c) What it means for two elements in a group to be *conjugate*. Give an example in S_4.

d) *Cyclic* group. Give an example of a cyclic subgroup of a non-cyclic group.

2)a) Define what it means for two groups to be isomorphic.

b) Prove that if G_1 is isomorphic to G_2, and G_1 has a nontrivial center then so does G_2 (non-trivial center here means that there is more than one element in the center).

c) Give an example of a non-trivial *automorphism* of S_3 (you do not need to prove that it is an automorphism).

d) How many different automorphisms of S_3, including the identity map, are there? Explain (but you do not need to illustrate all of them).

3)a) State the three Sylow theorems.

b)Use the Sylow theorems to show that every group of order 33 is abelian.

c)Use the Sylow theorems to answer the following question: Suppose that G is a non-abelian group of order 21. How many elements of order 3 must it contain?

d) Given a particular element of order three, explain how we know that it must be conjugate to at least 6 other elements of order three.

4)a) State the first isomorphism theorem.

b) Use this theorem and other group theory results to describe all possible groups that are homomorphic images of G other than G itelf or the trivial group $\{e\}$, assuming that the group G has 27 elements. In other words, assuming that G might be any group of order 27, describe all possible groups H such that there exists a homomorphism from G onto H, $H \neq \{e\}$ and H is not isomorphic to G. For each group H that IS possible, give an example of a group G and a homomorphism onto that particular H. You do not need to prove that your examples are homomorphisms.

5)a) Explain the class equation, using any non-abelian group as an example.

b) Use the class equation to prove that if G is a group of order 54 that has a trivial center (i.e. e is the only element in the center) then G must contain an element that commutes with exactly 27 elements.

Summer 2008 Algebra Comprehensive Examination, Part II

Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

1) Define the following terms and give one example of each.

a) *Irreducible polynomial* (give an example that is of degree 5 and irreducible over $\mathbf{Q}$).

b) *Unique Factorization Domain*

c) *Prime Ideal*

d) *Splitting field of a polynomial* (give TWO examples for this problem: one that IS the splitting field over $\mathbf{Q}$ of some irreducible polynomial of degree 3, and also give a field that has a root of this same polynomial but is NOT a splitting field).

2) Give examples of the following (and show why they are examples):

a) A field that is not the same field as $\mathbf{Q}(\sqrt[3]{2})$ but is isomorphic to it.

b) A maximal ideal in $\mathbf{Z}[x]$.

c) An irreducible polynomial whose splitting field is $\mathbf{Q}(\sqrt{2}, \sqrt{3})$. Explain why the degree of the polynomial you obtain makes sense by showing how to obtain the degree of the extension field $\mathbf{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbf{Q}$ in a completely different way.

3) a) Define the *kernel* of a ring homomorphism.

b) Define an *ideal* in a ring.

c) Show that in any ring R, a subset of R is an ideal iff it is the kernel of some ring homomorphism.

4) a) Explain how we know that the polynomial $x^3 + x + 2$ is irreducible in $\mathbf{Z}_3[x]$.

b) Let r be a root of this polynomial (in some larger field). Outline the proof that $\{a_0 + a_1 r + a_2 r^2 : a_0, a_1, a_2 \in \mathbf{Z}_3\}$ forms a field.

c) Does there exist a field with exactly 7 elements? exactly 8 elements? Give examples or explain.

5 a) Define what it means for a field extension K to be a *normal* extension of a field F.

b) Explain how we know that $\mathbf{Q}(\sqrt[5]{2})$ is not a normal extension of $\mathbf{Q}$.

c) Find the normal closure of $\mathbf{Q}(\sqrt[5]{2})$ over $\mathbf{Q}$ (i.e. find the smallest normal field extension of $\mathbf{Q}$ that contains $\mathbf{Q}(\sqrt[5]{2})$).

6) a) Define the *Galois Group* of a field K over a field F.

b) Give an example of a Galois group of finite order (other than 1) of some field K over $\mathbf{R}$.

c) Use the Fundamental Theorem of Galois Theory and Sylow's theorems to show that there is exactly one normal field extension of $\mathbf{Q}$ of order 4 that is contained in the field obtained as your answer to 5c) above.

Summer 2006 Algebra Comprehensive Examination, Part I

Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

1)Define the following terms:

a) The *cosets* of a subgroup in a group

b) What it means for two subgroups to be *conjugates*

c) *Sylow p* subgroup

d) The *normalizer* (or *centralizer*) of an element in a group

e) The *kernel* of a homomorphism

2)a) State the first isomorphism theorem.

b) Use this theorem to describe all possible homomorphic images of $\mathbf{Z}_{12}$.

3)a) Define what a *normal subgroup* is.

b) Show that if a subgroup N of a group G is normal then the set of cosets of N in G forms a group. Include a clear explanation of how the operation in this group is defined.

c) Explain clearly why the same process does not work for a subgroup H that is not normal in G. Use the subgroup $\{(1), (12)\}$ of S_3 as an example.

4)a) State the three Sylow theorems.

b) Suppose that G is a nonabelian group of order 363 $(3 \cdot 11^2)$. Determine the number of elements of G that have order 3 (with explanation).

5)a) Explain the class equation, using S_3 as an example.

b) Use the class equation to prove that every group of order 125 contains a non-trivial center.

6) Let G be a group and let x be any non-identity element of G. Define a function $f_x : G \to G$ by $f_x(g) = xgx^{-1}$ for every element $g \in G$.

a) Prove that f_x is an endomorphism (i.e. a homomorphism mapping onto itself).

b) Is it an automorphism as well? If so, prove that it is. If not, state an additional condition that would make it an automorphism.

Summer 2006 Algebra Comprehensive Examination, Part II

Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

1)Define the following terms:

a) Ring

b) Unique Factorization Domain

c) Prime Ideal

d) Homomorphism from one ring to another

e) Algebraic Field Extension

2)Give examples of the following (and show why they are examples):

a) A ring with no unity and no zero divisors.

b) A finite field.

c) An integral domain that is not a Euclidean ring.

3)a) Define the *splitting field* of a polynomial $p(x)$ over a field F.

b) Find the splitting field of the polynomial $x^5 - 2$ over $\mathbf{Q}$.

c) Find the degree of the splitting field from part b) above over $\mathbf{Q}$.

4)a) Explain how we know that the ideal $(x^4 - 2x^3 + 6x^2 - 8x + 10)$ is maximal in $\mathbf{Q}[x]$.

b) Let r be a root of the polynomial from a). Outline the proof (including stating the results needed) that the set $\{a + br + cr^2 + dr^3 : a, b, c, d \in \mathbf{Q}\}$ is a field.

5a) Define the *Galois Group* of a field K over a field F.

b) Let K be the splitting field of $x^3 - 2$ over $\mathbf{Q}$. Find the Galois Group of K over $\mathbf{Q}$.

c) Explain how the Fundamental Theorem of Galois Theory can be used to show that $\mathbf{Q}(e^{i2\pi/3})$ is the only field properly containing $\mathbf{Q}$ and properly contained in K that is itself a splitting field over $\mathbf{Q}$.

6) Let R be a commutative ring with unity, and let I be a prime ideal in R. Prove that the quotient ring R/I is an integral domain. Your proof should focus on showing that the operations of addition and multiplication are well-defined, and on showing that there are no zero divisors.

Summer 2004 Algebra Comprehensive Examination, Part I

Please answer as many of the following questions as is possible in the time allotted. Although you may not have time to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation), and *ability* to write correct proofs.

1. Define the following terms.
   a. The alternating group on *n* elements (A_n)
   b. What it means for two elements in a group to be *conjugates.*
   c. *p-group* and *Sylow* p-subgroup
   d. *Conjugacy class* in a group
   e. *Simple* group

2. (a) State *Lagrange's* Theorem.
   (b) Illustrate it with an example.

3. (a) State the three Sylow Theorems.
   (b) Suppose that G is a simple group of order 168. Determine the number of elements of G that have order 7 (with explanation).

4. (a) What is the *class equation* of a group? (State it and explain it.)
   (b) Illustrate the class equation in a nonabelian group of order 8.
   (c) Explain how the class equation might be used to show that two groups of the same order are not isomorphic.

5. Prove that for a given subgroup N of a group G, the following are equivalent:
   (a) $aN=Na$ for all a in G
   (b) $aNa^{-1} \subset N$
   (c) N is the kernel of a homomorphism $f: G \to H$

6. (a) Find the subgroup of S_5 generated by the set {(134),(13)}
   (b) Can there exist a homomorphism from S_5 onto this subgroup (in part (a))? If so, explain how it might be found/constructed, and if not, explain why not.

7. Consider the following abelian groups of order 36:

Z_{36}	$Z_{18} \oplus Z_2$	$Z_9 \oplus Z_4$
$Z_6 \oplus Z_6$	$Z_9 \oplus Z_2 \oplus Z_2$	$Z_3 \oplus Z_{12}$
$Z_4 \oplus Z_3 \oplus Z_3$	$Z_3 \oplus Z_3 \oplus Z_2 \oplus Z_2$	

   a. State exactly which of these are isomorphic to one another.
   b. Choose any two that are *not* isomorphic. Explain how you know (i.e. prove) that they are not.
   c. Choose any two that *are* isomorphic. Explain how you know that they are.

Summer 2004 Algebra Comprehensive Examination, Part II

Please answer as many of the following questions as is possible in the time allotted. Although you may not have time to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions completely), and *ability* to write correct proofs.

1. Define the following terms.
   a. *Integral Domain*
   b. *Euclidean Ring*
   c. *Unique Factorization Domain*
   d. *Field*
   e. *Degree* of an extension field E over a field F.

2. Give examples of the following (and show why they are examples):
   a. An integral domain that is not a Euclidean ring.
   b. An integral domain that is not a principal ideal domain.
   c. A non-commutative ring.

3. The set R/I consists of elements of the form $a+I$, where a is in R and I is an ideal in R. We define addition and multiplication as follows:

   $(a+I)+(b+I)=a+b+I$
   $(a+I)(b+I)=ab+I$

   (a) Prove that these operations are well-defined.
   (b) Replace I with an arbitrary subring S in the definitions above. Are the operations still well-defined? Explain.

4. (a) Define the splitting field of a polynomial $p(x)$ over a field F.
   (b) Find the splitting field of $x^4 - 7$ over **Q.**
   (c) Find the degree of the splitting field from part (b) over **Q**.
   (d) Choose a (non-identity) element of the Galois group of this splitting field over **Q** and describe it by its action on the roots of the polynomial.

5. Consider $\mathbf{Q}[\sqrt[3]{2}]=\{a+b(\sqrt[3]{2})+c(\sqrt[3]{2})^2: a,b,c \text{ in } \mathbf{Q}\}$. Outline the proof (including stating the results needed) that this set is a field.

6. Give, if possible (and if not, explain why not) an example of a polynomial that is:
   (a) Irreducible over **Q** but reducible over **C**.
   (b) Irreducible over **R** of degree 4.
   (c) Irreducible over **Q** but reducible over **Z**.

Summer 2003 Algebra Comprehensive Examination, Part I

Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation), and *ability* to write correct proofs.

1. Define the following terms.
   a. *Center* of a group
   b. The *index* of a subgroup in a group
   c. *p-group* and *Sylow* p-subgroup
   d. *Automorphism* of a group
   e. *Quotient* group
   f. The *action* of a group G on a set S.

2. State the following:
   a. *Lagrange's Theorem*
   b. The *class equation* of a group (explain the notation).

3. State the first isomorphism theorem, and then explain its significance and some of its possible uses.

4. Prove that no group of order 12 can be simple.

5. Find all positive integers n for which there exists an onto homomorphism from $f{:}S_3 \to Z_n$. Include an explanation of your reasoning.

6. Let H be a subgroup of index 2 in G. Prove that H must be a normal subgroup.

7. Let $f{:}G \to H$ be a homomorphism of groups. Show that the ker f=<e> if and only if f is 1-1.

8. Consider the following abelian groups of order 40:

Z_{40}
$Z_{20} \oplus Z_2$
$Z_{10} \oplus Z_4$
$Z_{10} \oplus Z_2 \oplus Z_2$
$Z_5 \oplus Z_8$
$Z_5 \oplus Z_4 \oplus Z_2$
$Z_5 \oplus Z_2 \oplus Z_2 \oplus Z_2$

   a. State exactly which of these are isomorphic to one another.
   b. Choose any two that are *not* isomorphic. Explain how you know (i.e. prove) that they are not.
   c. Choose any two that *are* isomorphic. Explain how you know that they are.

Summer 2003 Algebra Comprehensive Examination, Part II

Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions completely), and *ability* to write correct proofs.

1. Define the following terms.
   a. *Integral Domain*
   b. *Principal Ideal Domain*
   c. *Transcendental extension* (of a field)

2. Give examples of the following (and show why they are examples):
   a. A polynomial which is reducible in **Z**[x], but irreducible in **Q**[x].
   b. A ring that has non-trivial zero divisors

3. a. Let R be an arbitrary ring. Define *unit*, *prime element*, and *associates* in R.
   b. Define what it means for a ring R to be a *unique factorization domain*.
   c. Find the units in **Z**[i], **Q**[x] and **Q**.

4. a. Let K be an extension field of a field F. Define the degree of K over F.
   b. Find a basis for Q($\sqrt{2}$) over Q.
   c. Find a basis for Q($\sqrt{2}$,i) over Q.

5. a. Define the splitting field of a polynomial p(x) over a field F.
   b. Find the splitting field of (x^4-3) over **Q.**
   c. Find the degree of the splitting field from part b over **Q**.

6. State the Fundamental Theorem of Galois Theory and explain its significance.

7. a. Let R be an arbitrary ring. Define *prime ideal* and *maximal ideal* in R.
   b. Consider the ring 2**Z** (the ring of even integers). Show that the ideal 4**Z** (the multiples of 4) is maximal but not prime in 2**Z**.
   c. Consider the ring **Z**[x]. Show that the ideal <2> (the ideal generated by the element 2) is prime but not maximal in **Z**[x]. (Hint: consider <2,x>.)