Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

Problem 1. *Define the following terms and illustrate each one with an appropriate example:*

(a) normal subgroup;

(b) quotient group;

(c) p-Sylow subgroup.

Problem 2.

(a) Define the **center** *of a group.*

(b) Define the **normalizer** *of a in G.*

(c) State and explain the class equation, using any non-abelian group to illustrate.

(d) Prove that if the order of a group is a power of a prime, then the group has a non-trivial center.

Problem 3. *State and prove Cauchy's theorem for abelian groups.*

Problem 4. *Prove or Disprove: If H is a subgroup of G, then $Hx = Hy$ if and only if $xy^{-1} \in H$.*

Problem 5. *Consider a tetrahedron (a regular triangular pyramid). Let F be the set of its faces, V the set of its vertices, and E the set of its edges, and let S be the union of these three sets. Let G be the set of rotational symmetries of the tetrahedrom, and let G act on S in the natural way.*

(a) Let $v \in V$ be a vertex of the tetrahedron. Explain what is meant by the orbit $O(v)$ of v under this action, and what this turns out to be in this case.

(b) Explain what is meant by the stabilizer G_v and what this turns out to be in this case.

(c) Explain how $\circ(G)$, $\circ(O(v))$, and $\circ(G_v)$ are related, and use this relationship to determine $\circ(G)$ [1].

(d) Let $e \in E$ and $f \in F$ be an edge and a face of the tetrahedron. Check your answer for part (c) by doing the same computation with e and f that you did with v.

Problem 6. *Construct the* addition *and the* multiplication *tables for the ring $\mathbb{Z}_7$.*

(a) Using these tables please explain why $\mathbb{Z}_7$ is a field.

(b) Again using the tables, please carefully explain how you can solve the equation $2x + 5 = 1$. Briefly note which properties of a field structure are used at each step. Then verify your answer.

(c) Carefully prove that, in any ring R with unity, $(-1_R) \cdot a = -a$. Illustrate the meaning of this result in $\mathbb{Z}_7$.

Problem 7. *Let R be a commutative ring with unity.*

(a) Give the definition of an ideal *of R. What is the importance of this notion?*

(b) Define a maximal *ideal. Prove (directly, without using other theorems) that in a PID, if j is irreducible, then $\mathcal{J} = \langle j \rangle$ is a maximal ideal.*

[1] Here $\circ(S)$ denotes the order of a set S

(c) Let $\mathfrak{m} \triangleleft R$ be a maximal ideal. Prove (directly, without using other theorems) that $R/\mathfrak{m}$ is a field.

(d) Illustrate your proof by finding the inverse of $[13]$ in the ring $\mathbb{Z}_{31}$ following the steps in your proof in part (c).

Problem 8. *Let R and S be commutative rings with multiplicative identities and let $\varphi : R \to S$ be a ring homomorphism. Let $Q \triangleleft S$ be an ideal in S.*

(a) Show that $P = \varphi^{-1}(Q)$ is an ideal in R.

(b) If Q is prime, is P prime?

(c) If Q is maximal, is P maximal?

(d) If Q is principal, is P principal?

Note: *make sure to carefully define all of the terms here.*

Problem 9. *Consider the polynomial $p(x) = x^4 - 2 \in \mathbb{Q}[x]$.*

(a) Prove that this polynomial is irreducible *over $\mathbb{Q}$.*

(b) Construct, purely algebraically (i.e., using polynomial rings and not complex numbers) the splitting field of $p(x)$.

(c) Use it to find the order of the Galois group G of $p(x)$. Which familiar group is it isomorphic to? Can you construct the isomorphism explicitly?

(d) Give two non-trivial examples of the Galois subgroup/subfield correspondence.