Comprehensive Examination in Analysis
Ph.D. in Educational Mathematics
University of Northern Colorado
Summer 2017

Please answer as many of the following questions as you can in the time allotted. Although you may not be able to answer all the questions, passing will require you to show breadth of knowledge (answer a variety of questions), depth of understanding, (answer some questions that require explanation) and ability to write correct proofs.
The UNC honor code applies to this exam. You may not consult any books, notes, online resources, computing devices, or other aids during this exam, nor collaborate with any other person.

1. (a) Define what it means for a sequence of functions defined on $[0,1]$ to converge *pointwise* to a function on $[0,1]$, and what it means for a sequence of functions defined on $[0,1]$ to converge *uniformly* to a function on $[0,1]$.

   (b) Suppose that each of the functions $f_n(x)$ is continuous on $[0,1]$ and that the sequence $\langle f_n(x)\rangle$ converges uniformly to the function $f(x)$ on $[0,1]$. Prove that $f(x)$ is continuous on $[0,1]$.

   (c) Give an example of a sequence of continuous functions on $[0,1]$ that converges pointwise to a discontinuous function on $[0,1]$, and explain how you know that your example is correct.

2. (a) Suppose that f is a continuous real-valued function defined on the interval $[3,7]$. Prove the following special case of the extreme value theorem: f attains a maximum value on this domain, i.e. there exists a $c \in [3,7]$ such that $f(c) \geq f(x)$ for all $x \in [3,7]$.

   (b) Give an example to show that this conclusion is not true if we change the interval to $[3,7)$.

   (c) Give an example to show that this conclusion is not true if the interval stays $[3,7]$ but the function f is discontinuous.

   (d) Explain how to generalize your proof from (a) to prove the following version of the extreme value theorem: if K is a compact set in $\mathbb{R}^n$, and g is a continuous real-valued function defined on K, then g attains a maximum value on K.

3. (a) Define *Lebesgue outer measure.*
   (b) What does it mean for a set to be *Lebesgue measurable*, and why is this concept important?
   (c) Show that the usual "middle thirds" Cantor set on $[0, 1]$ has Lebesgue measure zero.
   (d) Define what it means for a subset of the reals to be "nowhere dense."
   (e) Prove that for every $n > 1$, there exists a measurable subset $E_n \subset [0, 1]$ which is nowhere dense and has Lebesgue measure $m(E_n) > 1 - \frac{1}{n}$.

       (For partial credit, or as a warmup, prove that there exists a measurable set E which is nowhere dense and has nonzero Lebesgue measure.)
   (f) Let E_n be the sets from the previous part, and let $F = \bigcup_{n=1}^{\infty} E_n$. What is $m(F)$?
   (g) Is it possible to construct the sets E_n so that $F = [0, 1]$?

4. (a) State the dominated convergence theorem.
   (b) Sketch the proof of the dominated convergence theorem, using the monotone convergence theorem, Fatou's lemma, or another argument of your choice.
   (c) Prove that

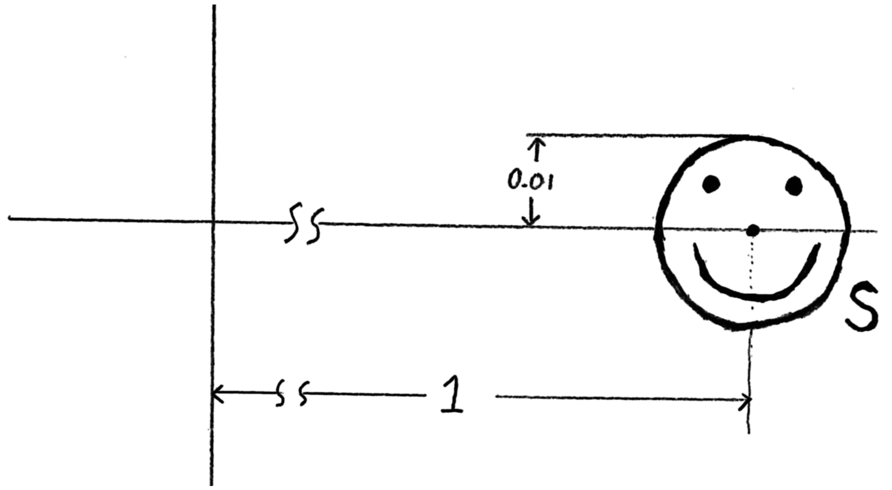
$$\lim_{n\to\infty} \int_0^1 x^n \sin(nx^2)\, dx = 0.$$

5. Suppose $A, B \subseteq \mathbb{C}$ are two connected sets such that $A \cap B \neq \emptyset$, and neither A nor B is contained in the other.

   (a) Must $A \cup B$ be connected? If yes, give a proof; if no, give a counterexample and prove that it works.
   (b) Must $A \cap B$ be connected? If yes, give a proof; if no, give a counterexample and prove that it works.
   (c) Must $A \setminus B$ be connected? If yes, give a proof; if no, give a counterexample and prove that it works.

6. (a) Let $f(z) = e^{i\pi z/4}$. Calculate $f(1)$ and $f'(1)$. Write your answers in rectangular form (i.e. $x + iy$).

   (b) Let $S \subset \mathbb{C}$ be the "smiley face" pictured below. The smiley face is centered at 1 and has a radius of 0.01.



   Using your answers from the previous part, make a labeled sketch of what we would expect the image of S under f to look like. Your sketch should make clear the location, size, and orientation of the image.

7. (a) State the Cauchy–Riemann equations and describe their significance.

   (b) Use the Cauchy–Riemann equations to prove that $f(z) = 1/z$ is analytic on $\mathbb{C} \setminus \{0\}$.

8. Let $\gamma : [0, 8\pi] \to \mathbb{C}$ be the curve defined by $\gamma(t) = 2e^{it}$. Use the Cauchy integral formula to compute

$$\int_\gamma \frac{z^2}{(z-6)(z-i)}\, dz.$$

9. (a) State some version of Cauchy's theorem (on contour integrals of analytic functions).

   (b) Let $f(z) = 1/z$. Let γ be a counterclockwise parametrization of the unit circle, and calculate $\int_\gamma f(z)\, dz$.

   (c) Explain why this does not contradict Cauchy's theorem.

10. (a) Find the first 5 terms of the Taylor series for $\mathrm{Log}(1 + z)$ about $z = 0$. (Here Log denotes the principal branch of the natural logarithm.)

   (b) What is the radius of convergence of this series? Justify your answer.

   (c) Consider the Taylor series for the function $f(z) = \dfrac{1}{1+z^2}$ about $z = 4 + 3i$. Use theorems about Taylor series and analytic functions to determine the radius of convergence of the series. You do not need to compute the series itself.