Comprehensive Examination in Analysis
Ph.D. in Educational Mathematics
University of Northern Colorado
Summer 2016

Please answer as many of the following questions as you can in the time allotted. Although you may not be able to answer all the questions, passing will require you to show breadth of knowledge (answer a variety of questions), depth of understanding (answer some questions that require explanation) and ability to write correct proofs.
The UNC honor code applies to this exam. You may not consult any books, notes, online resources, computing devices, or other aids during this exam, nor collaborate with any other person.

1. (a) Define what it means for a function defined on $(0,1)$ to be *continuous* and to be *uniformly continuous* on $(0,1)$.
   (b) Prove that if f is uniformly continuous on $(0,1)$ then the sequence $\left\langle f(\frac{1}{n^2})\right\rangle$ converges.
   (c) Give an example that shows that the result in part b) is false if we only assume that f is continuous and explain briefly.

2. (a) Let A, B be two closed subsets of $\mathbb{R}$. Suppose that $\inf\{|x-y| : x \in A, y \in B\} = 0$. Must A and B intersect? Prove or give a counterexample.
   (b) Let A, B be two compact subsets of $\mathbb{R}$. Suppose that $\inf\{|x-y| : x \in A, y \in B\} = 0$. Must A and B intersect? Prove or give a counterexample.

3. (a) Outline briefly the construction of Lebesgue measure on the real line, beginning with a definition of *outer measure*.
   (b) Show that the usual "middle thirds" Cantor set on $[0,1]$ has measure zero.
   (c) Define *Borel set*.
   (d) Sketch a proof of the following statement: If $A \subseteq [0,1]$ is Lebesgue measurable, then there exists a Borel set $B \supseteq A$ such that $m(A) = m(B)$. (Here m denotes Lebesgue measure).

4. (a) Let $f : [0,1] \to [0,7]$ be measurable. Show that for every $\epsilon > 0$ there exists a simple funtion $g : [0,1] \to \mathbb{R}$ such that $\int_{[0,1]} |f - g| \, dm < \epsilon$. Make it clear where we are using the fact that f is a measurable function.
   (b) Does there exist a sequence of functions $f_n : \mathbb{R} \to [0,\infty)$ converging pointwise to a function f, such that $\int f_n \, dm = 1$ for all n, but $\int f \, dm = 0$? Either disprove or find an example.
   (c) Does there exist a sequence of functions $f_n : \mathbb{R} \to [0,\infty)$ converging pointwise to a function f, such that $\int f_n \, dm = 1$ for all n, but $\int f \, dm = 2$? Either disprove or find an example.
   (d) Does there exist a sequence of functions $f_n : [0,1] \to [0,5]$ converging pointwise to a function f, such that $\int f_n \, dm = 1$ for all n, but $\int f \, dm = 0$? Either disprove or find an example.

5. (a) Define the terms "nowhere dense" and "meager".
   (b) State some version of the Baire category theorem.
   (c) Let $\{r_k : k \in \mathbb{N}\}$ be an enumeration of the rationals, and for each n define $U_n \subset \mathbb{R}$ by

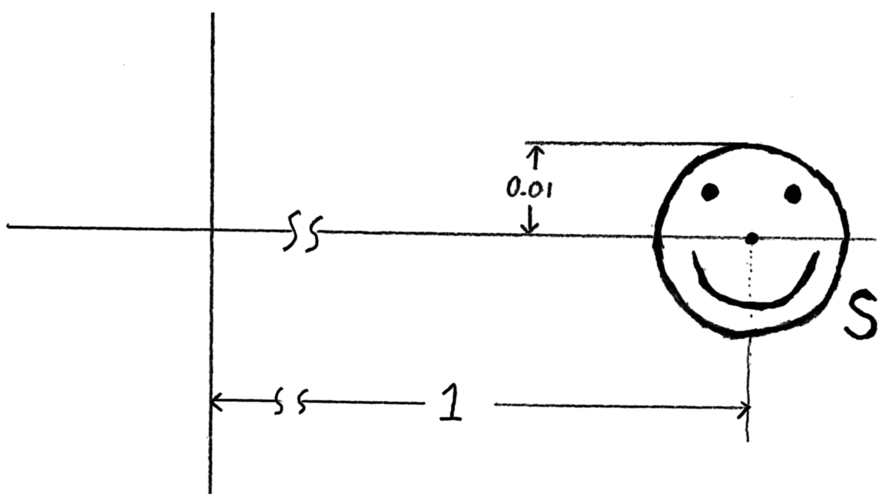
$$U_n := \bigcup_{k=1}^{\infty} (r_k - \frac{1}{n}, r_k + \frac{1}{n}).$$

   Use the Baire category theorem to show that the set

$$G = \bigcap_{n=1}^{\infty} U_n \text{ must contain some irrational numbers.}$$

   (Hint: In other words, show that $G^c \cup \mathbb{Q} \neq \mathbb{R}$.)

6. Let $S \subset \mathbb{C}$ be the "smiley face" pictured below. The smiley face is centered at 1 and has a radius of 0.01.



Suppose $f : \mathbb{C} \to \mathbb{C}$ is an analytic function such that $f(1) = 3+2i$ and $f'(1) = -1+i$. Based on this information, make a labeled sketch of what we would expect the image of S under f to look like. Your sketch should make clear the location, size, and orientation of the image.

7. (a) State the Cauchy–Riemann equations and describe their significance.
   (b) Suppose $f : \mathbb{C} \to \mathbb{C}$ is analytic at a point z_0. Using the Cauchy–Riemann equations, prove that $g(z) = f(z)^2$ is also analytic at z_0.
   (c) Let $h(z) = \overline{z}^2$. Determine, with proof, all points at which h is analytic.

8. Let $\gamma : [0, 6\pi] \to \mathbb{C}$ be the curve defined by $\gamma(t) = \cos(t) - 3i\sin(t)$. Use the Cauchy integral formula to compute

$$\int_\gamma \frac{1+z}{z^3 - 2iz^2 + 8z}\, dz.$$

9. (a) State some version of the Cauchy Differentiation Formula (also known as the "Cauchy integral formula for the derivative").

   (b) State Liouville's Theorem.

   (c) Outline how the Cauchy Differentiation Formula can be used to prove Liouville's Theorem.

10. (a) Find the Taylor series for $f(z) = \frac{1}{z}$ centered at $z = 1$.

    (b) Prove that the series converges normally on the open ball $\Delta(1, 1)$ centered at 1 with radius 1.

    (c) Prove that the series does not converge uniformly on $\Delta(1, 1)$.