Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

Problem 1.

(a) *Define what it means for a function defined on* $(0,1)$ *to be continuous and to be uniformly continuous on* $(0,1)$.

(b) *Let* $f : (0,1) \to \mathbb{R}$ *be uniformly continuous. Show that* f *can be defined at 1 in such a way that* f *is continuous on* $(0,1]$.

(c) *Give an example that shows that the result in part b) is false if we only assume that* f *is continuous, and explain briefly.*

(d) *Let* $\{r_n : n \in \mathbb{N}\}$ *be an enumeration of the rationals, and let* $f(x)$ *be the function from* $\mathbb{R}$ *to* $\mathbb{R}$ *defined by*

$$f(x) = \sum_{n: r_n < x} \left(\frac{1}{2^n}\right).$$

Show that the function f *is continuous at every irrational value of* x.

(e) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be continuous and suppose that* $f(3) = 2$ *and* $f(5) = 9$. *Show (without simply quoting the intermediate value theorem) that there exists a real number* $c \in (3,5)$ *such that* $f(c) = 7$.

Problem 2.

(a) *State and explain some version of the Baire Category Theorem. Your explanation should include a definition of "meager" sets.*

(b) *Use the Baire category theorem to explain how we know that the set of irrationals cannot be expressed as a countable union of closed sets.*

(c) *Is every set of measure zero a meager set? Justify your answer with a proof or counterexample.*

Problem 3.

(a) *If* F *is a closed subset of the interval* $[0,5]$, *explain from basic principles how to find the Lebesgue measure of* F.

(b) *Give an example of an uncountable set of measure zero, and explain how you know that your answer is correct.*

(c) *If* $f : [0,1] \to [0,1]$ *is a continuous function and* $E \subset [0,1]$ *has measure zero, must it be true that* $f(E)$ *has measure zero? Give a proof or a counterexample with some details.*

(d) *State and explain a precise version of the following statement: "every measurable set is nearly a Borel set".*

Problem 4.

(a) *Briefly discuss a few similarities and differences between the definition of the Lebesgue versus the Riemann Integral.*

(b) *Give a* ***specific*** *example involving integral convergence that works for Lebesgue Integrals but not for Riemann Integrals. Explain briefly (perhaps by quoting a major theorem) how we know that this specific convergence result works for Lebesgue Integrals and how we know that it does not work for Riemann Integrals.*

(c) *Show that if* $f : \mathbb{R}^n \to \mathbb{R}$ *is a non-negative measurable function and* $\int_{\mathbb{R}^n} f = 0$ *then* $f = 0$ *almost everywhere.*

(d) *Let* f *be an integrable function from* $[-1, 1]$ *to* $[-1, 1]$*. Show that for any* $\epsilon > 0$ *there exists a simple function* $s(x)$ *such that* $\int_{\mathbb{R}} |f(x) - s(x)|$*. Make sure to include a definition of "simple function".*

Problem 5. *Let* K *be a compact set in* $\mathbb{R}^n$*. Define the* diameter *of* K *to be*

$$\operatorname{diam}(K) = \sup_{p,q \in K} \rho(p, q),$$

where ρ *is the standard Euclidean distance function. Prove that there exist a pair of points* $p_1, p_2 \in K$ *such that* $\operatorname{diam}(K) = \rho(p_1, p_2)$*.*

Problem 6.

(a) *Give a concise construction of the field of complex numbers* $\mathbb{C}$ *(in particular, your answer should explain why the resulting algebraic structure is indeed a field).*

(b) *Explain the geometry of multiplication and division operations on complex numbers. Illustrate by some non-trivial examples.*

(c) *How do you extend the usual real function* $f(x) = \cos(x)$ *to a complex function* $f(z) = \cos(z)$*? How do you know that the resulting function is* differentiable*?*

(d) *How would you compute* $\cos(\mathfrak{i})$*?*

(e) *Find all* z *such that* $\cos(z) = \mathfrak{i}$*. Do it in two different ways: using a purely real-variable approach (work separately with the real and the imaginary parts), and then using a complex variable approach (i.e., inverse functions). Make sure to get the same answer.*

Problem 7.

(a) *Give a definition of what it means for a function* $f(z)$ *of a complex variable to be* differentiable *at a point* z_0*. Explain why* differentiability *is equivalent to* linearizability *of* $f(z)$*.*

(b) *Explain what the Cauchy-Riemann equations are.*

(c) *Find the derivative of a function* $f(z) = z^2 - 2z$ *at a point* z_0 *directly from the definition and by using the Cauchy-Riemann equations, then match your answers.*

(d) *Describe geometrically the local structure of the map* $f(z) = z^2 - 2z$ *at the point* $z_0 = \mathfrak{i}$*.*

(e) *Which part of the complex plane is* shrunk *and which one is* stretched *under the mapping* $f(z) = z^2 - 2z$*?*

Problem 8.

(a) *Describe all possible* Linear Fractional Transformations *that map the unit disk* $|z| < 1$ *onto itself. Note that there are many such maps. What additional parameters can be specified to make such a map* unique*?*

(b) *Find the explicit formula for such a mapping* w *such that* $w(1/2) = 0$ *and* $\arg w'(1/2) = \pi/2$*.*

Problem 9.

(a) *State* Cauchy's Integral Theorem *and* Cauchy's Integral Formula*. Give a sketch of a proof for both (this can be informal, just try to explain the main ideas).*

(b) *Use* Cauchy's Integral Formula *to evaluate*

$$\oint_C \frac{z^2 + 2e^{\pi z}}{z^3 - 3\mathfrak{i}z^2}\, dz,$$

where C *is a circle of radius* 1 *centered at* $\pi\mathfrak{i}$ *and oriented counter-clockwise. What changes if we take a circle of radius* 4 *(still centered at* $\pi\mathfrak{i}$*) instead?*