Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

Problem 1.

(a) *Define what it means for a function defined on* $(0, 1)$ *to be continuous and to be uniformly continuous on* $(0, 1)$.

(b) *Prove that if* $\{x_n\}$ *is a Cauchy sequence of points in* $(0, 1)$ *and* f *is uniformly continuous on* $(0, 1)$ *then* $\{f(x_n)\}$ *is a Cauchy sequence. Make sure to include the definition of Cauchy sequence.*

(c) *Give an example that shows that the result in part b) is false if we only assume that* f *is continuous and explain briefly.*

(d) *Define what it means for a function defined on* $(0, 1)$ *to be absolutely continuous on* $(0, 1)$ *(any equivalent definition is acceptable).*

(e) *Give an example of a function that is uniformly continuous on* $(0, 1)$ *but not absolutely continuous on* $(0, 1)$*, and explain briefly.*

Problem 2.

(a) *Explain from basic definitions how we know that a compact set in* $\mathbb{R}^n$ *must be closed and bounded (note that this is the easier direction of the Heine-Borel Theorem). Your answer should include a definition of compact and a definition of closed.*

(b) *Suppose that* K *is a compact subset of* $\mathbb{R}^n$, $f : K \to K$ *is continuous on* K *and* $\{x_n\}$ *is a sequence of points in* K *with the property that* $|f(x_n) - x_n| < \frac{1}{n}$. *Show that there exists a point* $x \in K$ *such that* $f(x) = x$. *Make it clear what results about compact sets you are using in your argument.*

(c) *Is the continuous image of every closed set closed (i.e. is it true that if* $f : \mathbb{R}^n \to \mathbb{R}^n$ *and* $F \subset \mathbb{R}^n$ *is closed then* $f(F)$ *is closed)? Explain briefly.*

(d) *Is the continuous image of every compact set compact? Explain briefly.*

Problem 3.

(a) *Define the following concepts, paying especial attention to the dependence/independence of your quantified variables: Outline briefly the construction of Lebesgue measure on the real line. For example we might want to first define it for open and closed bounded sets.*

(b) *Outline the construction of a set which has measure 1 on* $[0, 1]$ *but is meager on* $[0, 1]$. *Include a definition of meager.*

(c) *Give an example to show that it is not true that if* $A \subset [0, 1]$ *is measurable, then for any* $\epsilon > 0$ *there is a finite collection of intervals* $I_1, I_2, \ldots, I_n$ *such that* $A \subset \cup_{i=1}^{n} I_i$ *and*

$$\lambda\left(\left(\cup_{i=1}^{n} I_i\right) \setminus A\right) < \epsilon.$$

(d) *Despite your example to part (c) it is true that every measurable set "can be approximated" by a finite union of intervals in some sense. State a theorem about this and outline a proof.*

Problem 4.

(a) *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be bounded, measurable and positive, and let* E *be a bounded measurable subset of* $\mathbb{R}^n$*. Outline how we define the Lebesgue integral* $\int_E f$*. Make it clear where we are using the fact that* f *is a measurable function.*

(b) *Give an example of a sequence of measurable functions* $f_n : [0, 1] \to \mathbb{R}$ *such that the* f_n *converge to a function* f *pointwise on* $[0, 1]$ *but* $\int_{[0,1]} f_n$ *does not converge to* $\int_{[0,1]} f$.

(c) *Discuss and explain the dominated convergence theorem. Your example in part b) must not satisfy all the conditions (or else the integrals would have converged correctly). Discuss this briefly.*

(d) *Illustrate that the dominated convergence theorem does not work for Riemann integration by giving an example in which all the conditions are satisfied and yet* $\int_{[0,1]} f$ *does not exist.*

Problem 5.

(a) *State and explain some version of the Baire category theorem.*

(b) *Use the Baire category theorem to explain how we know that the irrationals are not a countable union of closed sets.*

Problem 6. *Give a definition of a* connected *set. Show that, if any two points of a set* E *belong to some connected subset of* E*, then* E *itself is connected.*

Problem 7.

(a) *Give a concise construction of the field of complex numbers* $\mathbb{C}$ *(in particular, your answer should explain why the resulting algebraic structure is indeed a field).*

(b) *Explain the operations of addition and multiplication of complex numbers from a geometric point of view.*

(c) *How do you extend the usual real function* $f(x) = \sin(x)$ *to a complex function* $f(z) = \sin(z)$*? How do you know that the resulting function is* differentiable*?*

(d) *How would you compute* $\sin(\mathfrak{i})$*?*

(e) *Find all* z *such that* $\sin(z) = \mathfrak{i}$*. Do it in two different ways: using a purely real-variable approach (work separately with the real and the imaginary parts), and then using a complex variable approach (i.e., inverse functions). Make sure to get the same answer.*

Problem 8.

(a) *Give a definition of what it means for a function* $f(z)$ *of a complex variable to be* differentiable *at a point* z_0*. Explain why* differentiability *is equivalent to* linearizability *of* $f(z)$.

(b) *Explain what the Cauchy-Riemann equations are.*

(c) *Find the derivative of a function* $f(z) = z^2 + 6z$ *at a point* z_0 *directly from the definition and by using the Cauchy-Riemann equations, then match your answers.*

(d) *Describe geometrically the local structure of the map* $f(z) = z^2 + 6z$ *at the point* $z_0 = 3\mathfrak{i}$.

(e) *Where is the map* $f(z) = z^2 + 6z$ *not conformal? What is the geometric structure of* $f(z)$ *at those points?*

Problem 9. *Consider the function* $f(z) = \dfrac{2z+3}{z^2+4}$ *and the mapping* $w = f(z)$ *given by this function (extended to the Riemann sphere* $\mathbb{CP}^1$*).*

(a) What is the degree of this map? How many preimages would a typical point $w \in \mathbb{CP}^1$ *have?*

(b) Find all images of $z = \infty$*. Find all preimages of* $w = \infty$*.*

(c) What are the points w *that have fewer preimages than generic points? What is happening there?*

Problem 10.

(a) State Cauchy's Integral Theorem *and* Cauchy's Integral Formula*. Give a sketch of a proof for both (this can be informal, just try to explain the main ideas).*

(b) Use Cauchy's Integral Formula *to evaluate*

$$\oint_C \frac{z-1}{(z^2+1)(z+1)^2}\,dz,$$

where C *is a circle centered at* $1+\mathrm{i}$ *and passing through the origin (oriented counter-clockwise).*