Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

Problem 1.

(a) *State (without proof) three equivalent ways to describe compactness of subsets of the Euclidean plane.*

(b) *Prove that two of these descriptions are equivalent. (You may choose which two.)*

(c) *Prove that the Cartesian product $K_1 \times K_2$ of two compact subsets $K_1, K_2 \subset \mathbb{R}$ is a compact subset of the plane.*

Problem 2. *Let D be a subset of the Euclidean plane..*

(a) *Explain what it means for D to be* connected*;* arcwise connected.

(b) *Prove that if D is* arcwise connected*, then it is* connected.

(c) *Prove that if D is open, then the converse statement is also true.*

(d) *Give an example of D that is connected but not arcwise connected.*

Problem 3.

(a) *Define the following concepts, paying especial attention to the dependence/independence of your quantified variables:*

- *uniform continuity of a function f;*
- *uniform convergence of a sequence of functions $f_n \overset{\text{unif}}{\longrightarrow} f$.*

(b) *Prove, or disprove by giving a counter-example: if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both uniformly continuous maps, then the composite function $f \circ g$ is uniformly continuous.*

(c) *Prove, or disprove by giving a counter-example: if $f_n : \mathbb{R} \to \mathbb{R}$ and $g_n : \mathbb{R} \to \mathbb{R}$ are functions that converge uniformly to limit functions f and g respectively as $n \to \infty$, and if f is uniformly continuous, then the composite functions $f_n \circ g_n$ converge uniformly to $f \circ g$ as $n \to \infty$. (Do not assume that f_n or g_n are necessarily continuous.)*

Problem 4. *State and prove the* Intermediate Value Theorem *for a real-valued function f whose domain is a subset of the real line.*

Problem 5.

(a) *List the axioms that define a σ-algebra.*

(b) *Define the Borel σ-algebra.*

(c) *Prove that every Lebesgue-measurable subset M of the real line is contained in a Borel set B such that $B \backslash M$ has measure zero. (Make sure to define what it means for a subset of the real line to have Lebesgue measure zero).*

Problem 6. *Outline the steps in the construction of a fat Cantor set: a closed subset $K \subset [0,1]$ with no interior points that has positive Lebesgue measure.*

Problem 7.

(a) *Give a concise construction of the field of complex numbers $\mathbb{C}$ (in particular, your answer should explain why the resulting algebraic structure is indeed a field).*

(b) *Explain, in some detail, why the multiplication by a complex number z with $|z| = 1$ corresponds geometrically to a rotation of the complex plane.*

(c) *Explain the meaning of 2^{i}. In particular, how many points on the complex plane does this expression describe? What about i^2?*

(d) *Solve the equation $z^3 = \mathrm{i}$ using both the complex analysis approach and the real variables approach (and make sure to explain why you get the same answers).*

Problem 8.

(a) *Give a definition of what it means for a function $f(z)$ of a complex variable to be differentiable at a point z_0.*

(b) *Explain what the Cauchy-Riemann equations are.*

(c) *Carefully prove that the Cauchy-Riemann equations give a necessary and sufficient condition for the differentiability of $f(z)$.*

(d) *Find the derivative of a function $f(z) = z^3$ at a point $z_0 \neq 0$ directly from the definition and by using the Cauchy-Riemann equations, then match your answers.*

Problem 9. *Consider the function $f(z) = z^2 - 2z$ and the mapping $w = f(z)$ given by this function (extended to the Riemann sphere).*

(a) *Describe all points at which the mapping given by $f(z)$ is* conformal.

(b) *What happens at the points where $f(z)$ is not conformal?*

(c) *Which part of the complex plane is* shrunk *and which part is* stretched *under this mapping?*

(d) *Find the angle of rotation and the local magnification factor of this mapping at the point $z = \mathrm{i}$.*

(e) *How many pre-images would a typical point on the w-plane have (give an example)? Which points are exceptional in that respect?*

Problem 10.

(a) *Prove that, if f is continuous in a neighborhood of the origin $z = 0$,* $\lim_{r\to 0} \int_0^{2\pi} f\left(re^{\mathrm{i}\theta}\right)\, d\theta = 2\pi f(0)$.

(b) *Let $C_r(a)$ denote the positively-oriented circle of radius r centered at a. Using the* Cauchy Integral Formula, *evaluate the following integrals.*

$$\oint_{C_3(\mathrm{i})} \frac{1}{z^2+9}\, dz \qquad \text{and} \qquad \oint_{C_3(\mathrm{i})} \frac{1}{(z^2+9)^2}\, dz$$