Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

Problem 1. *Give a definition of a* connected *set (if you prefer, you can assume that your sets are subset of the complex plane). Prove that if A and B are connected and $A \cap B \neq \varnothing$, then $A \cup B$ is connected. What about $A \cap B$?*

Problem 2.

(a) *Give a definition of what it means for a function $f(z)$ of a complex variable to be differentiable at a point z_0.*

(b) *Explain what the Cauchy-Riemann equations are.*

(c) *Carefully prove that the Cauchy-Riemann equations give a necessary and sufficient condition for the differentiability of $f(z)$.*

(d) *Find the derivative of a function $f(z) = z^{-1}$ at a point $z_0 \neq 0$ directly from the definition and by using the Cauchy-Riemann equations, then match your answers.*

(e) *Write down the Cauchy-Riemann equations using the polar coordinates in the domain of f.*

(f) *Can you use the polar form of the Cauchy-Riemann equations to find the derivative of the distance function $f(z) = |z^2| = r^2$?*

Problem 3.

(a) *Discuss the phenomena of multi-valuedness for functions of a complex variable. Why is it necessary? How do we work with it? How is this situation different from the real-variable case?*

(b) *Find all values of the following:*

- $(1 - \mathrm{i})^{1+\mathrm{i}}$;
- $2^{\sqrt{3}}$;
- $\mathrm{i}^{2/3}$ *(how many distinct values are there? Can you sketch them on the complex plane?)*

Problem 4.

(a) *Explain the notion of a conformal map.*

(b) *Consider the function $f(z) = z^2$ on the Riemann Sphere $\mathbb{CP}^1$. Show that this function defines a conformal map at the point $z = \mathrm{i}$. Describe the local structure of this map.*

(c) *What are the points where $f(z)$ fails to be conformal? What happens at those point?*

Problem 5.

(a) *Explain what are* Linear Fractional Transformations *and show that they form a group.*

(b) *Explicitly describe the subgroup of this group consisting of* LFT*s that permute the points 0, i, and ∞.*

(c) *Explicitly verify that composition and inverse operations for some elements of this group (one non-trivial example of each).*

Problem 6. *Let $f(z)$ be a continuous (but not necessarily analytic) function defined in the neighborhood of $z = 0$ and let C_r be a circle of radius r around the origin in the complex plane. Carefully prove that*

$$\frac{1}{2\pi} \lim_{r \to 0} \oint_{C_r} \frac{f(z)}{z} \, dz = f(0).$$

Problem 7.

(a) Find all roots of the polynomial $p(z) = z^4 + 4z^2 + 1$ and sketch their locations in the complex plane.

(b) Evaluate

$$\int_0^\infty \frac{dx}{x^4 + 4x^2 + 1}$$

using the Cauchy Integral Theorem. *Briefly, but carefully, justify the main steps in your reasoning.*