Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

Problem 1. *Let* (S, d) *be a metric space. Choose* $s_0 \in S$*. Show that* $f(s) = \text{dist}(s_0, s)$ *defines a uniformly continuous real-valued function* f *on* S*.*

Problem 2. *Let* f *be defined for all real* x*, and suppose that* $|f(x) - f(y)| \leq (x - y)^2$ *for all real* x *and* y*. Prove that* f *is constant.*

Problem 3. *For each of the following, either give a specific example, or tell why no example can exist.*

(a) A function that is continuous, but not uniformly continuous on the interval $[0, 1]$*.*

(b) A function f *that is continuous on* $[0, 1]$*,* $f(r) \leq 1$ *for all rational numbers* $r \in [0, 1]$*, but* $\sup_{x \in [0,1]} f(x) > 1$*.*

(c) A continuous function f *mapping* $(-1, 1)$ onto $(-1, 0) \cup (0, 1)$*. (*Remark*: A function* f *maps a set* E onto *a set* F *provided that* $f(E) = F$*.)*

True or False (Justify your claim or provide a counterexample)

(d) If $f : X \to Y$ *is a continuous function from a metric space* X *to a metric space* Y *and* $K \subseteq Y$ *is compact, then* $f^{-1}(K) \subseteq X$ *is compact.*

(e) If f *is a continuous real-valued function on* $X \subseteq \mathbb{R}$ *and* (x_n) *is a sequence in* X *such that* $f(x_n)$ *converges, then* (x_n) *converge.*

Problem 4. *(10 pts.) A real-valued function* f *on* $X \subseteq \mathbb{R}$ *is said to satisfy a* Lipschitz condition *if there exists a constant* M *such that* $|f(x) - f(y)| \leq M|x - y|$ *for all* x*,* y *in* X*. Show that Lipshitzian functions are uniformly continuous. Give an example of a uniformly continuous function that is not Lipshitzian.*