Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

Problem 1. *Define the following terms and illustrate each one with an appropriate example:*

(a) *normal subgroup;*

(b) *quotient group;*

(c) *p-Sylow subgroup.*

Problem 2. *Define the terms* left coset *and* partition. *Prove that if G is a group and H is a subgroup of G, then the left cosets of G form a partition of G.*

Problem 3.

(a) *Give an example of an order $n > 20$ such that there is exactly one group of order n up to isomorphism. Prove your answer.*

(b) *Give an example of an order $n > 20$ such that there are exactly two groups of order n up to isomorphism. Prove your answer.*

(c) *Give an example of an order $n > 20$ such that there are at least three groups of order n up to isomorphism. Prove your answer.*

Problem 4. *Prove that the only subgroup of D_5 that contains two different reflections is D_5 itself.*

Problem 5. *Construct the* addition *and the* multiplication *tables for the ring $\mathbb{Z}_{11}$.*

(a) *Using these tables please explain why $\mathbb{Z}_{11}$ is a field.*

(b) *Again using the tables, please carefully explain how you can solve the equation $3x + 7 = 5$. Briefly note whuch properties of a field structure are used at each step. Then verify your answer.*

(c) *Is it possible to solve a quadratic equation $2x^2 + 3x + 4 = 0$ in this field? Why or why not?*

Problem 6. *Let R be a commutative ring with unity.*

(a) *Give the definition of an* ideal *of R. What is the importance of this notion?*

(b) *Define a* maximal *ideal. Let $\mathfrak{m} \lhd R$ be a maximal ideal. Prove (directly, without using other theorems) that $R/\mathfrak{m}$ is a* field.

(c) *Illustrate your proof by finding the inverse of $[x+3]$ in the ring $\mathbb{Q}[z]/\langle x^2+5\rangle$ following the steps in your proof in part (b).*

Problem 7. *Let R be a commutative ring with unity.*

(a) *Define* irreducible *and* prime *elements. What is the importance of these definitions?*

(b) *Is it true that if an element is* prime *then it is* irreducible*? (Either give a proof or provide a counterexample).*

(c) *Is it true that if an element is* irreducible *then it is* prime*? (Either give a proof or provide a counterexample).*

(d) *Show that in a PID those properties are equivalent.*

Problem 8. *Consider the polynomial* $p(x) = x^3 + 2 \in \mathbb{Q}[x]$.

(a) Prove that this polynomial is irreducible *over* $\mathbb{Q}$.

(b) Construct, purely algebraically (i.e., using polynomial rings and not complex numbers) the splitting field of $p(x)$.

(c) Use it to find the order of the Galois group G *of* $p(x)$. *Which familiar group is it isomorphic to? Can you construct the isomorphism explicitly?*

(d) Give two non-trivial examples of the Galois subgroup/subfield correspondence (preferrably one normal and one not normal and explain what this means).