Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

Problem 1. *Define the following terms, and illustrate each one with an appropriate non-trivial example:*

(a) *right coset (for groups);*

(b) *p-group and p-Sylow subgroup;*

(c) *group homomorphism.*

Problem 2. *State and prove Lagrange's Theorem. Illustrate Lagrange's Theorem using a group of your choice.*

Problem 3.

(a) *Explain why the cosets of a group may not always form a group themselves under the natural operation inherited from the group, using an appropriate example.*

(b) *Define a condition under which the cosets do form a group, and prove that they do always form a group under this condition. Illustrate this with an appropriate example.*

Problem 4. *Let p be a prime. Prove that every p-group has a non-trivial center.*

Problem 5. *Consider a cube. Let F be the set of its faces, V the set of its vertices, and E the set of its edges, and let S be the union of these three sets. Let G be the set of rotational symmetries of a cube, and let G act on S in the natural way.*

(a) *Let $v \in V$ be a vertex of the cube. Explain what is meant by the orbit $O(v)$ of v under this action, and what this turns out to be in this case.*

(b) *Explain what is meant by the stabilizer G_v and what this turns out to be in this case.*

(c) *Explain how $\circ(G)$, $\circ(O(v))$, and $\circ(G_v)$ are related, and use this relationship to determine $\circ(G)$.*

(d) *Let $e \in E$ and $f \in F$ be an edge and a face of the cube. Check your answer for part (c) by doing the same computation with e and f that you did with v.*

(e) *How many subgroups of G are conjugate to G_v? Prove your answer.*

Problem 6. *General properties of **rings**.*

(a) *Give the definition of a ring. Carefully prove that a (non-zero) ring is* never *a group w.r.t. the multiplication operation (make sure to explicitly state all of the rings axioms that you are using in your proof).*

(b) *Give the definition of an* integral domain. *Give an example of an integral domain consisting of 4 elements, and then give an example of a commutative ring with unity that has four elements but is* not *an integral domain.*

Problem 7. *Give the definition of a field. Construct (by providing the addition and the multiplication tables), or explain why it is not possible to do, fields that have 3, 4, 5, and 6 elements.*

Problem 8. *Let R be a commutative ring with unity.*

(a) *Give the definition of an* ideal *of R. What is the importance of this notion?*

(b) Define a maximal *ideal. Let* $\mathfrak{m} \triangleleft R$ *be a maximal ideal. Prove (directly, without using other theorems) that* $R/\mathfrak{m}$ *is a* field. *Give an example.*

(c) Is it true or false that if a ring has a maximal ideal, then this ideal is unique?

Problem 9. *Consider the polynomial* $p(x) = x^4 - 3 \in \mathbb{Q}[x]$.

(a) Prove that this polynomial is irreducible *over* $\mathbb{Q}$.

(b) Construct, purely algebraically (i.e., using polynomial rings and not complex numbers) the splitting field of $p(x)$.

(c) Use it to find the order of the Galois group G *of* $p(x)$.

(d) Without computing G*, carefully prove, using the Sylow Theory, that* $G \simeq \mathcal{D}_4$ *(the dihedral group). Carefully state the Sylow theorems that you have used in your proof.*