Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

Problem 1. *Give examples of the following and show why they are examples OR explain why no such example exists.*

(a) *An integral domain with exactly 4 units (invertible elements).*

(b) *A ring that is not an integral domain with exactly 4 units.*

(c) *A polynomial of degree 3 that is irreducible in* $\mathbb{Z}_3[x]$

(d) *A polynomial of degree 3 that is irreducible over some field (i.e. in some* $F[x]$*) but not over a larger field (specify the polynomial and the two fields).*

(e) *A polynomial of degree 3 that is irreducible over* $\mathbb{Q}[x]$ *but not over* $\mathbb{Z}[x]$.

(f) *A subring that is not an ideal in* $\mathbb{Q}[x]$.

(g) *A subring that is not an ideal in* $\mathbb{Z}$

Problem 2.

(a) *Give a definition of a* ring homomorphism *and the* kernel *of a ring homomorphism.*

(b) *Give a definition of an* ideal *and a* maximal ideal *in a ring.*

(c) *Let* R *be a commutative ring and let* I *be an ideal in* R. *Explain from basic principles and with an example why multiplication is well-defined in* R/I.

(d) *Let* D *be a an integral domain and* I *be an ideal in* D. *Show from basic principles (i.e. don't use a big theorem that makes this easy) that every non-zero element in* D/I *has an inverse iff* I *is a maximal ideal.*

(e) *Explicitly show the multiplication for* $\mathbb{Q}[x]/(x^2-5x+6)$, *i.e. letting* $I=(x^2-5x+6)$, *show what* $(ax+b+I)(cx+d+I)$ *is equal to in the quotient ring.*

(f) *Are there any zero divisors in the quotient ring* $\mathbb{Q}[x]/(x^2-5x+6)$*? Explain.*

Problem 3.

(a) *Outline the proof that the set* $\mathbb{Q}(\sqrt[3]{2}) = \{a + b(\sqrt[3]{2}) + c(\sqrt[3]{2})^2 : a, b, c \in \mathbb{Q}$ *is a field. Your proof should include the defining of a homomorphism* ϕ *from* $\mathbb{Q}[x]$ *onto* $\mathbb{Q}(\sqrt[3]{2})$.

(b) *What specific element in* $\mathbb{Q}(\sqrt[3]{2})$ *is* $\phi(x^4 - 2x^3 + x^2 + 4x + 1)$, *assuming* ϕ *is the same homomorphism you defined when you answered part a) above.*

(c) *Define the* degree *of a field extension (using the concept of dimension) and state and explain what the degree of the extension field* $\mathbb{Q}(\sqrt[3]{2})$ *over* $\mathbb{Q}$ *is.*

(d) *Is* $\mathbb{Q}(\sqrt[3]{2})$ *a splitting field of any polynomial over* $\mathbb{Q}$*? Explain why or why not. If not, say what element could be adjoined to make it a splitting field.*

(e) *The complex number* c *is not equal to* $\sqrt[3]{2}$ *but* $\mathbb{Q}(\sqrt[3]{2})$ *is isomorphic to* $\mathbb{Q}(c)$. *Find all possible values of* c *and explain.*

Problem 4. 1. *Find or explain how to construct a field of order 6 or explain why none exists.*

2. *Find or explain how to construct a field of order 7 or explain why none exists.*
3. *Find or explain how to construct a field of order 8 or explain why none exists.*

Problem 5.

(a) *Give a definition of the* Galois group *of a field* $\mathbb{K}$ *over a field* $\mathbb{F}$.

(b) *Let* G *be the Galois group of the splitting field of* $x^4 - 3$ *over* $\mathbb{Q}$. *Identify this group in more familiar group notation of your choice.*

(c) *Find the subgroup of* G *in your notation that corresponds to the field* $\mathbb{Q}(\sqrt{2}i)$ *via the correlation given by the Fundamental Theorem of Galois Theory, and explain.*

(d) *Find the field extension in the lattice that is not a splitting field over* $\mathbb{Q}$, *correlate it with the appropriate subgroup, and explain the concept from the fundamental theorem that this illustrates.*