Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

Problem 1.

(a) Give a definition of a ring*.*

(b) Give a definition of an integral domain*.*

(c) Give a definition of a field*.*

(d) Discuss the similarities and difference between these three mathematical objects.

Problem 2. *Give examples of the following (and show why they are examples):*

(a) An integral domain that is not a unique factorization domain.

(b) An ideal in a ring that is prime but not maximal (make it clear what ring you are using for this example).

(c) An ideal that is not principal in some ring (make it clear what ring you are using for this example).

(d) A field $\mathbb{F}$ with the property that there exists only one (i.e., trivial) automorphism of $\mathbb{F}$.

(e) A field $\mathbb{F}$ that has only one non-trivial *automorphism.*

Problem 3.

(a) Give a definition of a ring homomorphism *and a* kernel *of a ring homomorphism.*

(b) Is every subring of the integers the kernel of a ring homomorphism? Explain.

(c) Is every subring of $\mathbb{Q}[x]$ the kernel of a ring homomorphism? Explain.

(d) Discuss the relationship in general between subrings and kernels of ring homomorphisms.

Problem 4. *Let ϕ be the homomorphism that sends $\mathbb{Q}[x]$ into the real numbers $\mathbb{R}$ by evaluation at $\sqrt[3]{5}$, $\phi(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = a_0 + a_1 \sqrt[3]{5} + a_2(\sqrt[3]{5})^2 + \cdots + a_n(\sqrt[3]{5})^n$.*

(a) Is this homomorphism surjective (onto)?

(b) Describe its kernel K (make sure to prove that your answer is correct).

(c) ***Explicitly*** *describe the quotient ring $\mathbb{Q}[x]/K$. Your answer should be in terms of the way that quotient rings are constructed and not dependent on more advanced theorems.*

(d) Using your description in part (c), find the image of $2+3x+5x^2+7x^3$ under the quotient homomorphism.

(e) Relate your answer to (c) to the answer we obtain from the fundamental isomorphism (homomorphism) theorem for rings.

Problem 5.

(a) Give a definition of the splitting field *of a polynomial $p(x)$ over a field $\mathbb{F}$.*

(b) Give a definition of the degree *of a field extension.*

(c) Find the degrees of each of the following fields over $\mathbb{Q}$:

$$\mathbb{Q}[\sqrt{5} + \mathrm{i}] \qquad \mathbb{Q}[\sqrt{5}, \mathrm{i}] \qquad \mathbb{Q}[\sqrt{5}\mathrm{i}].$$

(d) Are any of the three fields in part (c) above equal to each other (i.e. literally the same field)? Explain.

Problem 6.

(a) Give a definition of the Galois group *of a field* $\mathbb{K}$ *over a field* $\mathbb{F}$.

(b) Suppose that $p(x)$ *is a an polynomial of degree 5 in* $\mathbb{Q}[x]$ *with the property that its splitting field has degree* 120 *over* $\mathbb{Q}$. *Use the Fundamental Theorem of the Galois Theory and Sylow's theorems to show that there is a field extension of* $\mathbb{Q}$ *of degree 15 over* $\mathbb{Q}$.

(c) Is the field $\mathbb{K}$ *from part (b) also a splitting field of some polynomial over* $\mathbb{Q}$. *Explain.*

(d) With $\mathbb{E}$ *as in part (b), explain how we know that any subfield of* $\mathbb{E}$ *that is itself a splitting field of some polynomial over* $\mathbb{Q}$ *must be of degree* 2 *over* $\mathbb{Q}$.