Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

Problem 1. *Define the following terms, and give an appropriate example (you do not need to prove that your examples are correct).*

(a) *A* kernel *of a homomorphism from one group to another. Give an example of a homomorphism from* $\mathbf{D}_4$ *to some other group that has a kernel of order* 2*.*

(b) *An* automorphism *of a group. Give an example of a nontrivial (not the identity map) automorphism of* $\mathbf{S}_3$*.*

(c) *What it means for two subgroups of a group to be* conjugate*. In* $\mathbf{S}_4$*, describe subgroups* $\mathbf{H}_1$*,* $\mathbf{H}_2$*, and* $\mathbf{H}_3$ *such that* $\mathbf{H}_1$ *and* $\mathbf{H}_2$ *are conjugate, and* $\mathbf{H}_1$ *and* $\mathbf{H}_3$ *are not, and briefly explain.*

(d) *The* normalizer *(or* centralizer*) of a group. Let* g *be the element of* $\mathbf{D}_4$ *that corresponds to (if we think about* $\mathbf{D}_4$ *as a symmetry group of a unit square on the plane centered at the origin) reflection about the* y *axis. What is the normalizer of this element?*

Problem 2.

(a) *Define what it means for two groups to be* isomorphic*, give an example.*

(b) *Prove that if* $\mathbf{G}_2$ *is isomorphic to* $\mathbf{G}_1$*, and* $\mathbf{G}_1$ *has an element of order* 4*, then so does* $\mathbf{G}_2$*.*

(c) *Is the result in part (b) still true if we replace "isomorphic to" with "is the homomorphic image of"? If it is true, prove it. If it is false, give an example.*

Problem 3.

(a) *State the three Sylow theorems.*

(b) *Use the Sylow theorems to show that if* $\mathbf{G}$ *is a group of order* 30 *then it has either a normal subgroup of order* 3 *or a normal subgroup of order* 5*.*

(c) *Suppose that* $\mathbf{G}$ *is a group of order* 24*. Use the Sylow theorems to describe the possible number of subgroups of order* 3 *that* $\mathbf{G}$ *might have. Find examples that show that these possibilities actually occur.*

Problem 4.

(a) *State the* First Isomorphism Theorem*.*

(b) *Explain the theorem in detail using a homomorphism from* $\mathbf{A}_4$ *onto* $\mathbb{Z}_3$ *as an example.*

(c) *Use this theorem and other group theory results to show that if* $\mathbf{G}$ *is a group of order* 100 *that has a homomorphism onto* $\mathbb{Z}_5 \times \mathbb{Z}_5$ *then* $\mathbf{G}$ *is abelian.*

Problem 5.

(a) *Explain the class equation using* $\mathbf{S}_3 \times \mathbb{Z}_2$ *as an example.*

(b) *Use the class equation to prove that every group of order* 81 *has a non-trivial center.*