Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

Problem 1.

(a) Give a definition of a group*.*

(b) Give a definition of a ring*.*

(c) Give a definition of a field*.*

(d) Discuss the similarities and difference between these three mathematical objects.

Problem 2. *Give examples of the following (and show why they are examples):*

(a) A principal ideal in $\mathbb{Q}[x]$*.*

(b) Is there an example of an ideal in $\mathbb{Q}[x]$ *that is not principal? If so, give one. If not, prove that no such ideal exists.*

(c) An ideal that is not principal in some ring (make it clear what ring you are using for this example).

Problem 3.

(a) Give a definition of a ring homomorphism*.*

(b) Give a definition of a kernel *of a ring homomorphism.*

(c) Give a definition of a subring *of a ring.*

(d) Give an example of a subring that is not the kernel of any homomorphism (again, make it clear what ring you are working in).

(e) What properties distinguish kernels of homomorphisms from other subrings?

(f) Is every subring with the additional properties you have listed in part (e) a kernel of some ring homomorphism? Explain.

Problem 4. *Let* ϕ *be the homomorphism that sends* $\mathbb{Q}[x]$ *into the complex numbers* $\mathbb{C}$ *by evaluation at* $\mathfrak{i}$*,* $\phi(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = a_0 + a_1\mathfrak{i} + a_2\mathfrak{i}^2 + \cdots + a_n\mathfrak{i}^n$*.*

(a) Is this homomorphism surjective (onto)?

(b) Describe its kernel K *(make sure to prove that your answer is correct).*

(c) ***Explicitly*** *describe the quotient ring* $\mathbb{Q}[x]/K$*.*

(d) Using your description in part (c), find the image of $2 + 3x + 5x^2$ *under the quotient homomorphism.*

(e) Relate your answer to (c) to the answer we obtain from the fundamental isomorphism (homomorphism) theorem for rings.

Problem 5.

(a) Give a definition of the splitting field *of a polynomial* $p(x)$ *over a field* $\mathbb{F}$*.*

(b) Give a definition of the degree *of a field extension over another field.*

(c) Are the degrees of the splitting fields of the polynomials $x^3 - 1$ *and* $x^3 - 3$ *the same over* $\mathbb{Q}$*? Explain.*

(d) *Find the degrees of each of the following fields over* $\mathbb{Q}$*:*

$$\mathbb{Q}[\sqrt{3}+\sqrt{6}] \qquad \mathbb{Q}[\sqrt{3}\cdot\sqrt{6}] \qquad \mathbb{Q}[\sqrt{3},\sqrt{6}].$$

(e) *Are any of the three fields in part (d) above equal to each other (i.e. literally the same field)? Explain.*

Problem 6.

(a) *Give a definition of the* Galois group *of a field* $\mathbb{K}$ *over a field* $\mathbb{F}$*.*

(b) *Suppose that* $p(x)$ *is a an polynomial of degree 5 in* $\mathbb{Q}[x]$ *with the property that its splitting field has degree* 40 *over* $\mathbb{Q}$*. Use the Fundamental Theorem of the Galois Theory and Sylow's theorems to show that there is a field extension of* $\mathbb{Q}$ *of degree 4 that is normal over* $\mathbb{Q}$ *and contained in the splitting field of* $p(x)$*.*