Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

Problem 1. *Define the following terms, and give an appropriate example:*

(a) *A* homomorphism *from one group to another. Give an example of a homomorphism from* $\mathbf{S}_3$ *to some* $\mathbf{Z}_n$*, where* $n \neq 1$.

(b) *What it means for two subgroups to be* conjugate. *Give an example of two different subgroups of* $\mathbf{D}_4$ *that are conjugate to each other.*

(c) *A* Sylow p-subgroup. *Give an example of a Sylow 2-subgroup of* $\mathbf{A}_4$.

(d) *An* automorphism *of a group. Give an example of a non-trivial automorphism of* $\mathbf{Z}_6$.

Problem 2.

(a) *Define what the* order *of an element of a group means.*

(b) *Prove that if* G_1 *is isomorphic to* G_2*, and* G_1 *has an element of order 3 then so does* G_2.

(c) *Is there an element of order 15 in* $\mathbf{S}_8$*? Is there an element of order 14? Explain briefly.*

Problem 3.

(a) *State the three Sylow theorems.*

(b) *Use the Sylow theorems to prove that every group of order 45 is abelian.*

(c) *Use the Sylow theorems to find the number of elements of order 2 in a non-abelian group of order 14. Explain.*

Problem 4.

(a) *State the first isomorphism theorem.*

(b) *Explain the theorem in detail using a homomorphism from* $\mathbf{D}_4$ *onto* $\mathbf{Z}_2 \times \mathbf{Z}_2$ *as an example.*

(c) *Use this theorem and other group theory results to prove carefully that there is no homomorphism from* $\mathbf{S}_4$ *onto* $\mathbf{Z}_4 \times \mathbf{Z}_2$.

Problem 5.

(a) *Prove that the set of all automorphisms of a group is itself a group (with composition being the operation). You do not need to prove associativity of the composition operation.*

(b) *Consider the following statement: the set of all automorphisms of a group is isomorphic to the group itself. Is this statement true for all groups? Is it false for all groups? Explain as clearly as possible and with examples.*

Problem 6. *Explain the class equation using* $\mathbf{S}_3 \times \mathbf{Z}_2$ *as an example.*