VISUALIZATION ON CONES AND POOL TABLES USING GEOMETER'S SKETCHPAD

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ABSTRACT: In this paper, I will discuss two related activities that I have developed for use in a college geometry course directed at pre-service secondary and elementary school teachers. These activities ask students to develop geometry on pool tables and on cones. The two activities are closely related, and both ask students to relate computer models with physical models. The pool table activity was developed as a lead in for the cone project, which is by far the most successful project that I have used for getting pre-service teachers actively engaged with the full cycle of exploring a new space, making conjectures about it, and then coming up with explanations for and proofs of the conjectures.

KEYWORDS: College Geometry, Pre-service teachers, Geometer's Sketchpad, visualization, cones, pool tables, covering spaces.

1 INTRODUCTION

In this paper, I will discuss two related activities that I have developed for use in a college geometry course directed at pre-service secondary and elementary school teachers. These activities ask students to develop geometry on pool tables and on cones. The two activities are closely related, and both ask students to relate computer models with physical models. The pool table activity was developed as a lead in for the cone project, which is by far the most successful project that I have used for getting pre-service teachers actively engaged with the full cycle of exploring a new space, making conjectures about it, and then coming up with explanations for and proofs of the conjectures.

The first activity is a computer lab using Geometer's Sketchpad, in which students are asked to construct a model of a pool table, showing all of the paths that a cue ball can take before hitting another ball after bouncing off of a given number of rails. They do this by looking at all possible repeated reflections of the target ball over sides of the table. This leads naturally to a kind of covering space construction and a proof that there are $4n$ paths in which the ball hits exactly n rails. The other project comes later in the course, and asks students to write an open-ended paper in which they explore aspects of geometry on the cone on their own, in groups. One of the reasons that this assignment has been so fruitful is that there are many different directions in which students can take it, and several of these have related back to the pool table lab in ways that have sometimes surprised me. Students have used the ideas developed in the pool table lab to construct 2D covering space models of the cone in Sketchpad; printed and cut these out in order to make physical models of the cone; photocopied them onto transparencies to give a wonderful, physical representation of a covering space; and used their computer models to answer questions such as that of how many straight paths connect two points on a particular cone.

The main course in which I have used these activities has been a college geometry course at the University of Northern Colorado. Typically, half of the students in this class are math majors who intend to teach mathematics at the high school level, and half of the students are pre-service elementary school teachers who have chosen a concentration in mathematics. The course is taught in a modified Moore method, discovery-learning centered style. It includes a great deal of writing, and is loosely based around the materials in chapters 1–4, 6, and 7 in the book *Experiencing Geometry: Euclidean and non-Euclidean with History* [2], although the course has no official textbook that the students must buy. Most of the materials used in the course are available on the web at `http://www.mathsci.unco.edu/course/math341/`.

The two activities described in this paper take up a significant portion of this course: the pool table activity described in Section 2 typically takes a week of class time to complete, and the cone project described in Section 3 typically takes three weeks of class time. I have found that the investment of time is well worth it. These activites are deeply connected to and give students significant experience with three of the four NCTM content standards for geometry, which state that students should learn to:

- analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships;
- apply transformations and use symmetry to analyze mathematical situations; and
- use visualization, spatial reasoning, and geometric modeling to solve problems.

Just as importantly, these activities give students an unparalleled opportunity to make mathematical discoveries themselves, to develop their own ideas, and to communicate these ideas to their peers. Thus, they also give students an extended experience with the NCTM problem solving, reasoning and proof, and communications process standards.

2 GEOMETRY ON POOL TABLES

The first activity that I want to discuss asks students to look at geometry on a pool table. It asks them to find a way to predict the path a ball will take after bouncing off of a rail, using Geometer's Sketchpad; and then to use this to figure out what direction to aim a ball to hit another ball after one bounce off of a rail; after 2 bounces; and after 3 bounces. Finally, students are asked to determine a formula $f(n)$ for the number of different paths that a cue ball can take before hitting another ball after n bounces.

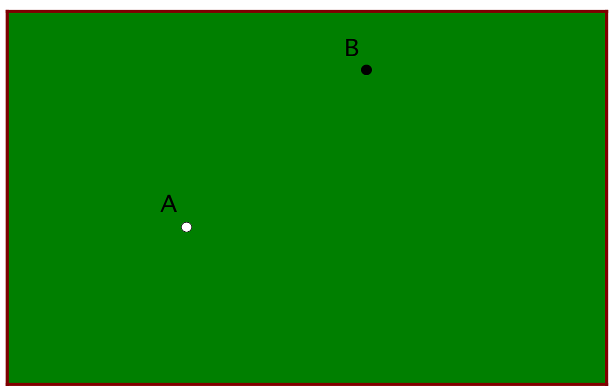

Figure 1. A Sketchpad Pool table. How can we construct the path that the white cue ball A will take after being hit in the direction of point B?

Figure 1 shows a pool table constructed in Geometer's Sketchpad. The picture reproduced here is in black and white, but students seem to derive a surprising amount of pleasure from making their constructed tables green, so that they look more like real pool tables. The first problem that we want to consider is how to construct the path that the white cue ball A will take when hit towards another point B, and, in particular, the path it follows after it bounces against the top rail.

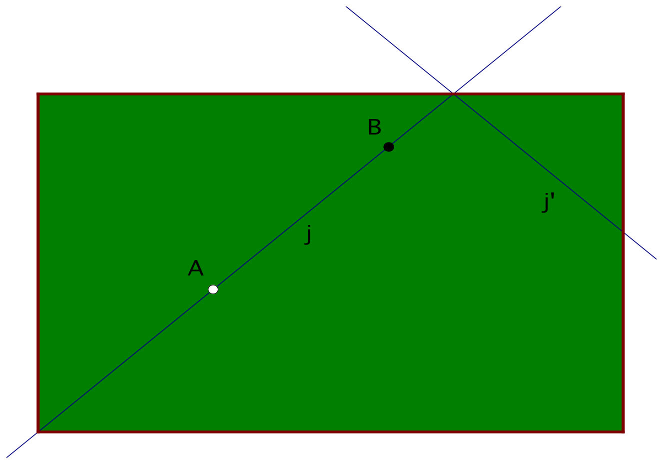

Figure 2. j' is the reflection of j over the top rail of the table, and shows the path the ball takes after bouncing.

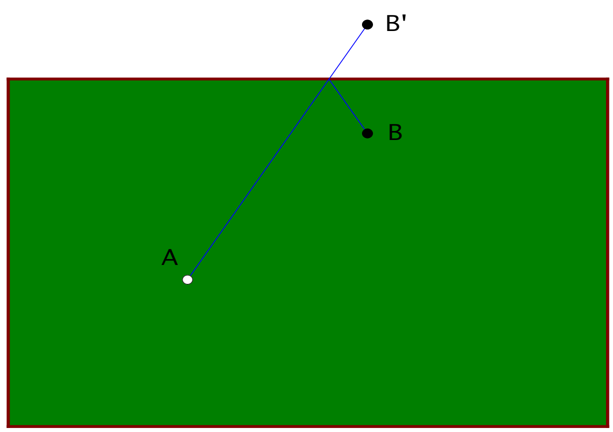

Figure 3. To construct a path that hits B after one bounce, aim for B'.

There are several different ways we could solve this problem, but the most useful way to do it is also one of the least obvious: draw the line connecting the two balls, and then reflect that line over the top rail, as shown in Figure 2, in which line j' is the reflection of line j over the top rail. Students can show that this makes the angle of incidence congruent to the angle of reflection; this follows from the vertical angle theorem and the fact that reflections preserve angle measure.

Now, how can we aim the cue ball so that it bounces off of the top rail *before* hitting point B, which we now think of as representing the 8-ball? The trick is to use what we just figured out: if we want the line showing the initial path of the cue ball to hit B after it is reflected over the rail, we should aim it for the reflection B' of B over this rail, as illustrated in Figure 3. This method of aiming for a reflected ball can be found in some books on playing pool. ([3], for example.)

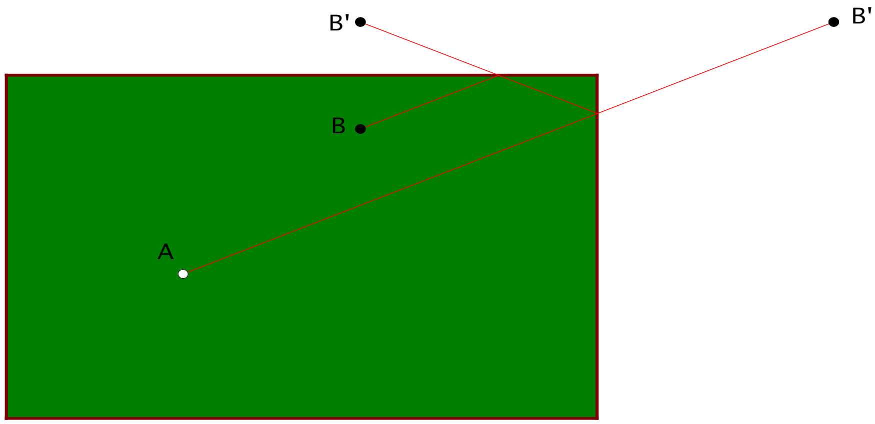

Figure 4. To construct a path that hits B after two bounces, aim for B''.

We can extend this idea further, to paths that bounce more than once. For example, Figure 4 shows a path in which the cue ball bounces twice before hitting the eight ball, constructed by aiming for the point B'' that was created by reflecting the point B' over one of the other sides of the table. The initial path, when reflected over that side, becomes a path aiming for B', which when reflected over the top rail becomes a path aiming for B. We can extend this idea to create paths that bounce any given number of times by reflecting the target ball that many times over the sides of the table.

Now that we know how to create paths that bounce any given number of times, we can ask the question: how many different paths are there that bounce n times? Many students initially guess that there should be $4 \cdot 3^{(n-1)}$ paths, because there are four choices for the first wall to bounce off of, and then three choices for each subsequent bounce, because the ball cannot bounce against the same wall twice in a row. This does give an upper bound for the possible number of paths, but it is too big: not all possible sequences of sides correspond to actual paths, depending on the locations of the balls. For example, in Figure 4, in the path shown, the cue ball bounces off of the right rail and then the top rail; but there is not any path that bounces off of the top rail and then the right rail.

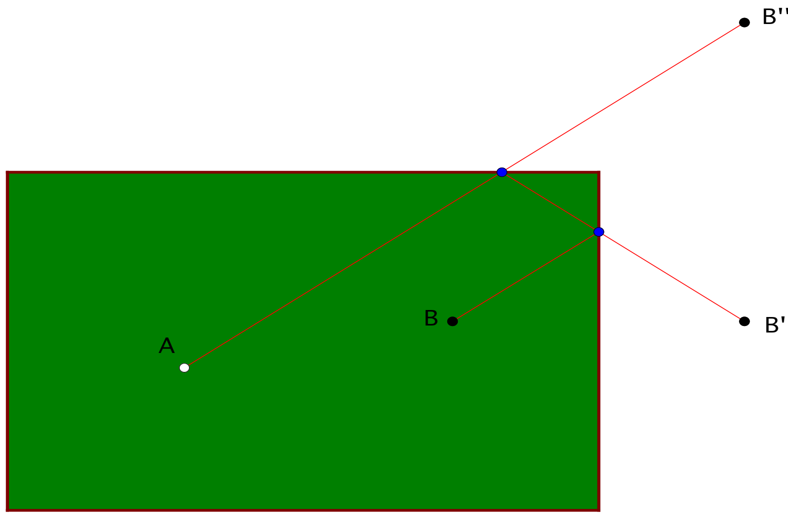

Figure 5. Another path that hits B after two bounces.

We can move the eight ball so that there will be a path that bounces off of the top rail and then the right rail, as shown in Figure 5, but then there will not be a path that bounces off of the right rail and then the top rail. Notice, though, that this path is still aiming for the point B'' obtained as before by two reflections of the point B. This same point can be obtained in two different ways, by reflecting over the top and right sides of the table in either possible order. Only one of these ways of getting B'' will correspond to an actual path, though. Which way it is will depend on where the balls are, and which wall the line connecting the cue ball to B'' crosses first.

This suggests a better way to count the number of possible paths with n bounces: count the image balls obtained by reflecting B n times over different walls. This idea gives rise to the grid of reflected pool tables shown in Figure 6. Our original table is in the center. There are four tables around the original table obtained by reflecting it over its four sides. This shows that there are exactly four different ways of aiming the cue ball to hit the eight ball with one bounce: we can aim for the reflection of the eight ball on any of these four tables. Likewise, there are eight tables that can be obtained from our original table by preforming two reflections; they form a diamond shape around the previous four tables. These eight tables give us exactly eight different ways of hitting the eight ball after two bounces, again by aiming for the eight doublely reflected images of the eight ball. This pattern continues: there are twelve tables obtained by reflecting the original table three times, again arranged in a diamond formation around the previous eight, so there are twelve possible paths that bounce three times. In general, each diamond will contain four more tables than the previous one, so there must always be $4n$ paths that reach the eight ball after bouncing n times. (We are ignoring the possible problems of pockets, corners, and balls in the way.)

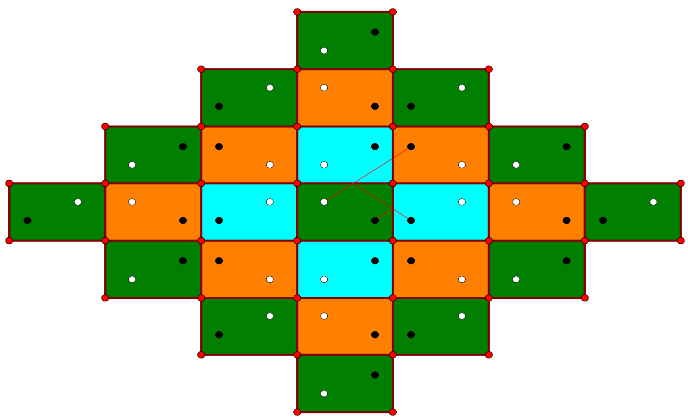

Figure 6. The grid of reflected pool tables.

In my class, we generally spend between two and three class periods, each an hour and fifteen minutes long, on this problem. In that time, in my experience, most students (with varying degrees of help from me) can figure out how to construct paths with up to three bounces, and some but not all of them will have figured out how to count the number of possible paths with n bounces.

It should be noted that the pool table, with lines given by the paths of the balls, is not quite a manifold, because it does not look like the plane at the edges; but what we have given is essentially a covering space construction.

3 CONE PROJECT

The preceding pool table project is fun, and the students enjoy it because it connects to a real world problem that they are interested in. However, it also serves as a perfect lead in to our final project.

This final project in the class asks students to explore geometry on the cone in groups of three. The project is very open-ended: students explore what interests them, write a report, and present their findings to their classmates. We spend a lot of time in class on this project—generally, three to four weeks. I have found that it has always been worth the investment in time; it has been the best project I have used for getting students at this level really doing mathematics themselves. It is based on the problems in chapter four of [2], but its approach is much more open ended, so that students feel like they are picking their own problems to explore in a subject they have not been told anything about in advance—*i.e.*, like they are mathematicians themselves.

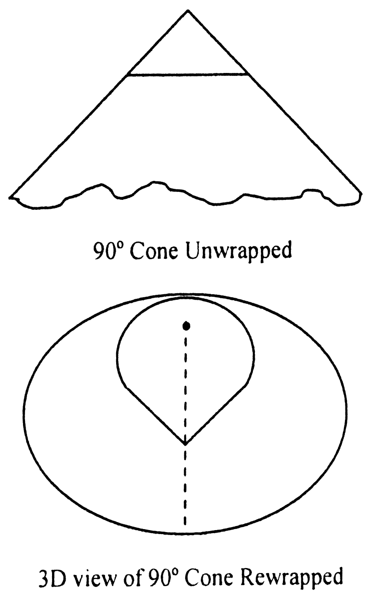

Figure 7. Illustration of a cone with a 90° cone angle.

In order for this assignment to make sense, I need to explain a little bit about geometry on the cone. The cones that we want to consider are infinite cones, that extend infinitely without a base. These cones have one special point, the tip, called the *cone point*, where the surface is not smooth. If we cut the cone open along a straight ray coming out of the cone point, we get the shape of an angle that we can lay flat; and we can measure the size of the cone by the size of this angle, called the *cone angle*. See Figure 7 for an illustration of this idea, taken from a student paper [1]. Notice that a straight line on the flattened cone becomes a straight line on unflattened cone when the sides of the angle are glued back together. Lines on the cone that do not pass through the cone point are straight if they are straight when the cone is flattened; and we will consider lines that pass through the cone point to be straight if they have half-turn symmetry—that is, if the two rays coming out of the cone point make an angle that is exactly half of the cone angle. Lines like these can be thought of as being the "straightest possible" lines through the cone point.

Geometry on the Cone

See how much geometry you can develop on cones with cone angle less than 360 degrees (*e.g.*, the 90 degree cone). You may also be interested in comparing these with cones with cone angle greater than 360 degrees (*e.g.*, the 450 degree cone) and/or cylinders. For any of these surfaces you can ask any of the following questions:

- What do the intrinsically straight lines look like? What happens to lines that run into the cone point?
- What do circles look like?
- Which of Euclid's Postulates are true?
- What is the sum of the interior angles of a triangle?
- What is the holonomy of a triangle?
- Which triangle congruence theorems hold (or do not hold)?
- Is there a unique straight line joining any two points? If not, for which points is there a unique straight line joining them?
- Do any straight lines intersect themselves? If so, how many times?

Explore any of these questions, or others, that interest you, and write a paper explaining what you find.

Figure 8. Text of the cone project assignment given to students.

Figure 8 shows the text of the cone assignment as given to the students. It consists mainly of a list of questions that the students can choose to investigate. Some students have written successful papers that mainly investigated a single question, while others have looked at several or even most of these questions. Some of the questions refer to the idea of holonomy, which may be an unfamiliar concept for many readers; holonomy will not be discussed further in this paper, but the interested reader can find its definition in chapter 7 of [2].

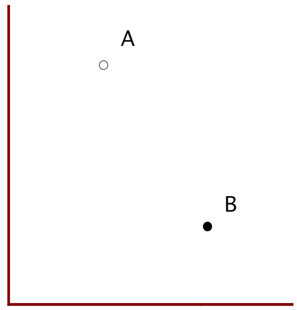

Figure 9. Two points on a flattened 90° cone.

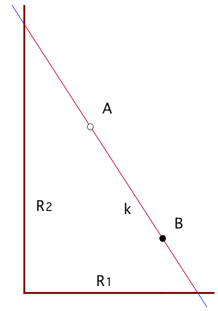

Figure 10. What happens if we extend line k on the cone?

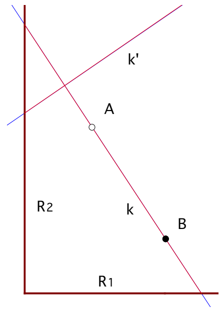

Figure 11. The line k' is formed by rotating k by 90° about the cone point.

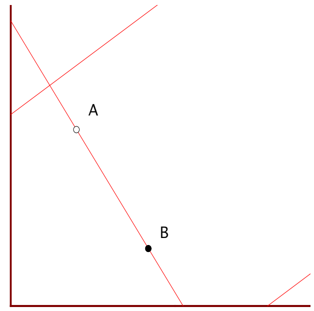

Figure 12. A complete line on the 90° cone.

Consider two points A and B on a cone with a 90° cone angle—that is, on a cone that can be sliced open and laid flat to form a 90° angle, as shown in Figure 9. We'd like to understand how to find the line that connects them on this cone. We can easily draw a line k between the two points, as shown in Figure 10, but we need to figure out what happens to the line after it hits one of the sides of our flattened cone. Notice how similar this is to the previous problem of predicting the path that a ball would take on a pool table after bouncing against a wall. In fact, we can imagine that our points on the cone are two-dimensional balls in a two-dimensional flatlander's game of pool on a cone—but without any walls to bounce off of. If we hit ball A towards ball B, ball B will continue along the same line until it reaches the bottom-right side of our flattened cone, labeled R_1. Since the cone is made by gluing R_1 to R_2, the path must continue from a point on R_2 that is the same distance from the cone point as the point where it hit R_1—we'll call the continuation of the path k'. Furthermore, the path must make the same angle with R_2 that it made with R_1. Since R_1 and R_2 form a 90° angle, this means that k' must be rotated 90° from k. So the easiest way of finding the continuation of the path k is by rotating it by 90° about the cone point, as shown in Figure 11. If we use this method of rotation to extend the line in both directions, and then erase the parts of the lines that do not lie within our flattened cone, we get a picture of a line on the cone like the one shown in Figure 12. Students can easily figure out how to make pictures like this using Geometer's Sketchpad, and can then cut them out and tape them together to make really nice three dimensional cones. My experience is that students in my class always start out by making paper models and drawing figures on them by hand, but once one group discovers that they can make more accurate models using Sketchpad, then several other groups will notice and also start using Sketchpad.

Once we know how to draw lines on cones, we can ask the question of how many different lines can be drawn between two given points. This is analogous to the question on the pool table of how many different ways the cue ball can be aimed to hit the eight ball after bouncing a given number of times. In that case, we aimed for reflected copies of our original balls, because the continuation of a line was produced by reflecting it. In the present case, the continuation of a line is produced by rotating it about the cone point. This suggests that we rotate our original points about the cone point and aim for the rotated points. This leads us to the grid of points shown in Figure 13. Notice that we have again formed a covering space. Because there are four copies of the black point B in our grid, there are four different ways to connect the white point to the black point on our 90° cone: the direct way that we already found, shown in Figure 14; one way directly across the back of the cone, shown in Figure 15; and two paths that wrap around the cone and cross themselves before hitting the other point, shown in Figures 16 and 17, one in each direction.

This should give the flavor of how this problem is related to the previous one, and how students can use these kinds of ideas to start exploring the properties of lines on the cone. Students can (and will, given the chance) explore many related questions using these kinds of methods: How many ways are there of connecting two points on a cone? How many times will a line cross itself? At what angles? Does this depend on where the points or lines are, or what the cone angle is? How?

These questions can all be explored either by hand on paper models, or using Sketchpad; what's more, once correct conjectures have been made, they almost all have accessible proofs that some students will find.

One question that is not addressed in [2] and that has turned out to be much more interesting than I expected it to be when I first asked it is: What do circles look like on cones? This is a question that is much easier to explore using Sketchpad than it is to explore by hand. It also turns out to be a question that students really like.

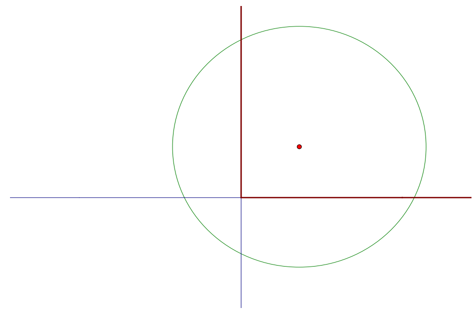

Figure 18. What happens to this circle on the 90° cone?

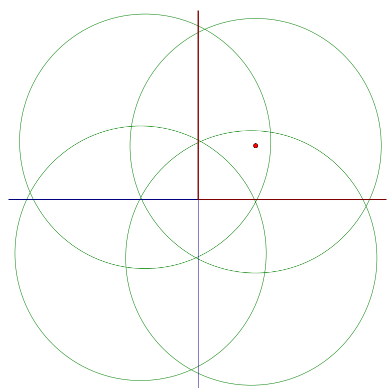

Figure 19. A circle on the 90° cone.

Most students classify circles on the cone into 3 sets:

1. Those whose center is at the cone point. These are the circles formed by the intersection of the cone with a horizontal plane; they are circles both extrinsically and intrinsically, although intrinsically, they contain fewer than 360 degrees.
2. Those whose center is far away from the cone point, so that the cone point lies outside the circle. These look exactly like planar circles, for

the most part, so they are not that interesting, although occasionally students will notice that it is possible for them to intersect themselves, and all of the intersections lie along a single line through the cone point.

3. Those whose center is close to, but not on, the cone point. These are the most interesting ones. If we start with a circle like the one shown in Figure 18, we can use the covering space idea to draw rotated copies of the circle, as shown in Figure 19. If we cut out one quadrant of this figure and tape the sides together, we will get a perfect model of a circle on a cone. I have had students that have drawn circles like these by hand, using string or a compass, but they're much easier to produce using Sketchpad.

One particularly nice illustration of the idea of a covering space and how to use it to draw circles was invented by a group of pre-service elementary school teachers in my class last semester [5]. They came up with idea of photocopying Sketchpad sketches like the one in Figure 19 onto overhead transparencies, for use in their in-class presentation. One of their models of a circle on a 90° cone is shown in Figure 20. As another example, Figure 21 shows one of the other transparencies that they made to show a circle on the 180° cone, and Figure 22 shows the same transparency rolled up into a cone. I think that models like this would be really useful to show in any class discussing covering spaces for the first time.

Figure 20. A student-produced covering space model of a circle on a 90° cone, made by photocopying a picture like that shown in Figure 19 onto a transparency.

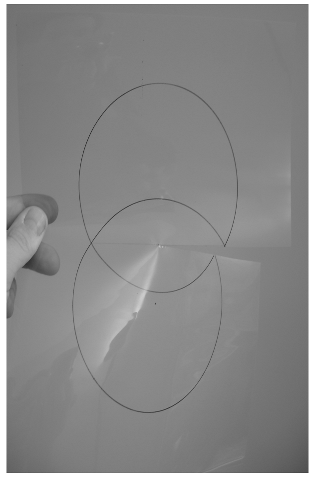

Figure 21. A student-produced covering space model of a circle on a $180°$ cone, unwrapped.

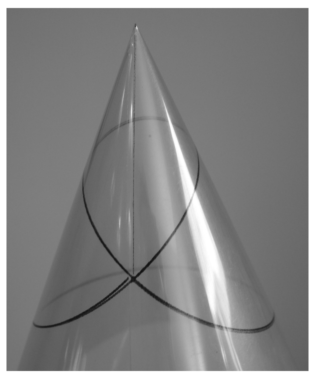

Figure 22. A student-produced covering space model of a circle on a cone, made by wrapping up the transparency in Figure 21.

A further question is: what do circles look like on cones whose cone angle does not divide evenly into $360°$? In this case, the plane is not a covering space. Most students that have looked at this question have come up with an answer like that shown in Figure 23, which is again taken from a student paper [1]. We can consider a circle to be the set of all points that can be connected to a given center by a straight line segment of a given length. A radius of the circle can therefore be drawn in any direction from the center point; because there are $360°$ around this point, there must be

360° worth of circle. In this particular case, we have a 270° cone, so we have 270° worth of circle before the circle intersects itself on the side of the cone directly opposite the center of the circle; since there are 90° of circle left over, we have 45° more on each side. There is one isolated point of the circle that is not connected to the others, which is created by the radius that goes directly through the cone point.

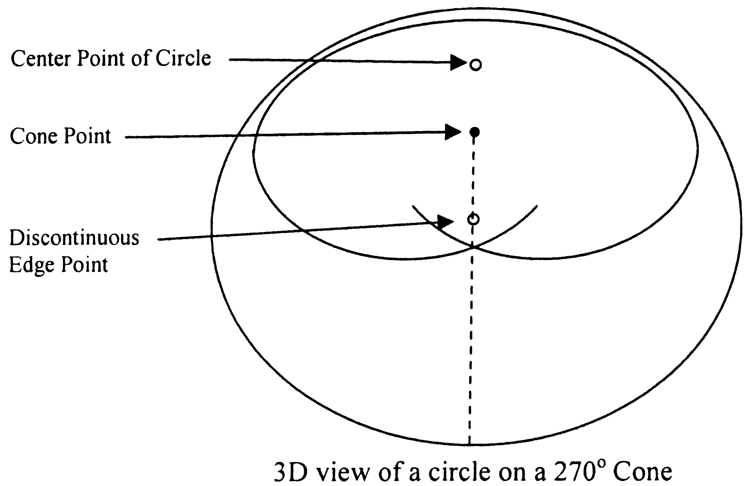

Figure 23. A circle on a cone whose cone angle does not divide evenly into 360°: Version 1.

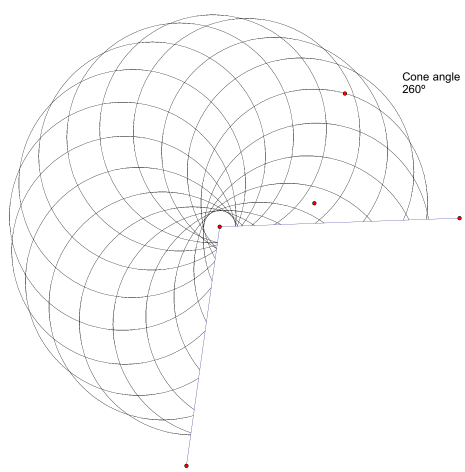

Figure 24. A circle on a cone whose cone angle does not divide evenly into 360°: Version 2.

Last semester, however, one group looking at this question came and showed me the picture in Figure 24, which they said was a picture of a circle on a flattened 260° cone, and told me that they had conjectured that the number of arcs in a circle like this was the least common multiple of the cone angle and 360°, divided by the cone angle [4]. I told them that I did not think this was right, having only seen examples like Figure 23 before. They persisted in saying that their example should be a circle, however, and we came to the conclusion that this could be a circle if you considered a circle to be what the students called "a continuous arc." In standard terminology, we would call this a curve of constant curvature—and, sure enough, their example is a curve of constant curvature—just like a circle on the plane. So which shape we consider to be a circle on the cone depends on which of the possible planar definitions of a circle we choose to generalize. These constant curvature circles are especially fun to look at on physical models of the cone; the way they match up feels surprising and cool.

There are a lot of reasons why I love doing this project with students: it builds on everything we have done all semester, and connects directly back to other assignments like the pool table assignment. It is far more open ended than most undergraduate projects, and students can find fruitful things to explore in many different directions (only some of which have been discussed here!). They get to make discoveries and conjectures on their own in a new area, which is an experience I think we need to give our students as much as possible; and then they can often prove their own conjectures. But one aspect that I particularly enjoy is that every semester, students show me new ideas that I have never seen before, like the transparency cone models and the constant-curvature cone circles that students showed me last semester.

The assignment is well summed up by the conclusion of one group's paper from last spring: "We had a great time with this project because of how much constructive learning took place. We were surprised at how far we were able to develop our thoughts and build off each other's ideas." [6] I'm continually surprised by this myself, and I think that more instructors should try giving this kind of assignment—they, too, might be surprised at how much real mathematics their students are capable of!

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BIOGRAPHICAL SKETCH

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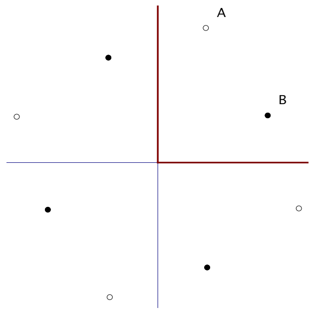

Figure 13. A grid of $90°$ cones.

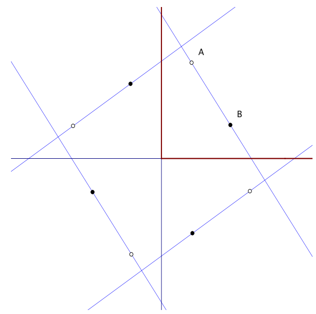

Figure 14. How to connect two points on the $90°$ cone: Method 1.

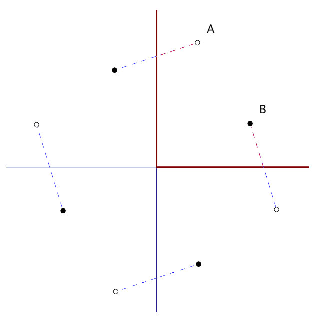

Figure 15. How to connect two points on the $90°$ cone: Method 2.

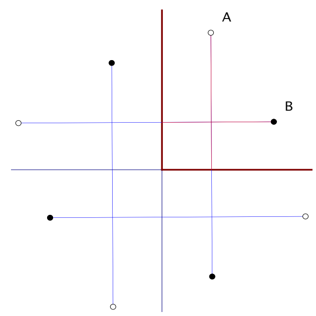

Figure 16. How to connect two points on the $90°$ cone: Method 3.

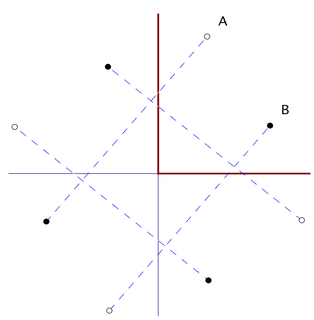

Figure 17. How to connect two points on the $90°$ cone: Method 4.