

## On the Inconsistency of Mumma's **Eu**

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**Abstract** In several articles, John Mumma has presented a formal diagrammatic system **Eu** meant to give an account of one way in which Euclid's use of diagrams in the *Elements* could be formalized. However, largely because of the way in which it tries to limit case analysis, this system ends up being inconsistent, as shown here. **Eu** also suffers from several other problems: it is unable to prove several wide classes of correct geometric claims, and contains a construction rule that is probably computationally intractable and that may not even be decidable.

### 1 Introduction

In several articles ([10], [8], and [9]), John Mumma presents a formal diagrammatic system **Eu** meant to give an account of one way in which Euclid's use of diagrams in the *Elements* could be formalized. More detail about how **Eu** works is found in his Ph.D. dissertation [7]. **Eu** is similar in many respects to my earlier diagrammatic formal system for Euclidean geometry **FG**, which is described in my book *Euclid and his Twentieth Century Rivals: Diagrams in the Logic of Euclidean Geometry* [6]. However, **Eu** differs from **FG** in a few key ways:

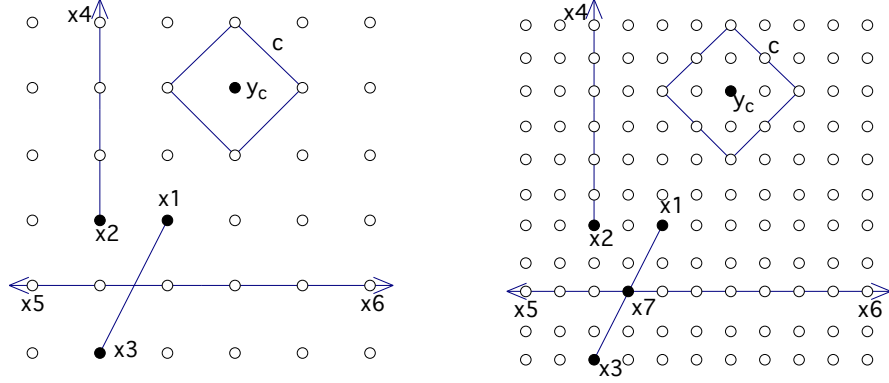
1. **Eu** treats diagrams as geometric objects, while **FG** treats diagrams as topological objects;
2. **Eu** requires that representations of line segments, rays, and infinite lines are actually straight, while **FG** does not;
3. **Eu** tries to limit the number of separate cases that must be considered when elements are added to existing diagrams, while **FG** provides a method of producing all of the topologically distinct diagrams that could result and requires that they all be considered; and
4. **Eu** divides proofs into two stages, a construction stage and a demonstration stage, with different rules, while **FG** makes no such distinction.

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**Figure 1** A diagram and its completion.

All of these differences are ultimately problematic for **Eu**, with the result that **Eu** is inconsistent, and suffers from a number of other problems which will be discussed here.

Mumma discusses the differences between **Eu** and **FG** from his point of view in an article in *Philosophia Mathematica* [9].

**Eu** shares properties 1–3 above with another proposed diagrammatic formal system for Euclidean geometry, Isabel Luengo’s **DS1**, which is described in a chapter of the book *Logical Reasoning with Diagrams* [3] and in her Ph.D. thesis [2]. **DS1** is likewise unsound, and suffers from many problems similar to those of **Eu**. A detailed account of the problems with **DS1** can be found in [6, Appendix C].

## 2 Equivalent Diagrams in **Eu**

For completeness, a quick summary of parts of **Eu** is given here, but an interested reader can find the full details in Mumma’s Ph.D dissertation [7].

In **Eu**, a syntactic diagram is a square array of evenly spaced dots which can be filled in to represent points, connected with straight line segments to represent line segments, rays, or infinite lines (depending on if they have arrows on their ends), and can be connected to form convex polygons that represent circles. Each object in a diagram can be labeled by a variable name so that we can refer to it. His system also includes metric assertions, which are conjunctions of equalities and inequalities that can refer to objects in diagrams via their labels. The left side of Figure 1 shows an example of a diagram; this diagram contains four points labeled  $x_1$ ,  $x_2$ ,  $x_3$ , and  $y_c$ ; one line segment,  $x_1x_3$ ; one ray,  $x_2x_4$ ; one infinite line,  $x_5x_6$ , and one circle  $c$  whose center is at point  $y_c$ .

Because each diagram is intended to represent a range of possible arrangements of circles, points, and lines in the Euclidean plane and several different diagrams may represent the same arrangements, Mumma then defines an equivalence relation on his diagrams. In order to do this, he first defines a completion of a diagram  $D$  to be a new diagram  $D'$  that contains the same geometric elements but such that the number of dots in the underlying square

array of dots is increased so that every intersection point between two geometric elements in the diagram occurs at a dot in the array and is added as a point to the diagram. As an example, the diagram on the right side of Figure 1 shows a completion of the diagram on the left. Notice that the number of dots in the array has been increased so that the intersection point of segment  $x1x3$  and line  $x5x6$  occurs at a dot, and that the new point  $x7$  has been added here. Also note that there would be further intersection points of objects in the diagram if the segment and ray were extended, but these intersection points are not added to the completion of the diagram under this definition.

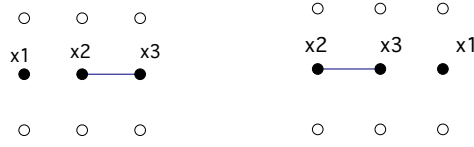
Having defined a diagram's completion, Mumma then defines diagram equivalence as follows:

**Definition 2.1** Two diagrams  $D_1$  and  $D_2$  are equivalent if there is a completion  $D'_1$  of  $D_1$ , a completion  $D'_2$  of  $D_2$ , and a bijection  $f$  between the points, lines, rays, segments, and circles of  $D'_1$  and  $D'_2$  induced by the labelings of each diagram, such that for all points  $P$  and  $Q$ , linear elements (lines, rays, or segments)  $M$  and  $N$ , and circles  $C_1$  and  $C_2$  in  $D'_1$ ,

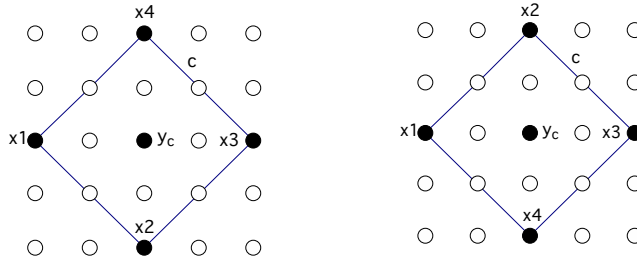
1.  $P$  lies on  $M$  in  $D'_1$  iff  $f(P)$  lies on  $f(M)$  in  $D'_2$ ;
2.  $P$  and  $Q$  lie on a given side of  $M$  in  $D'_1$  iff  $f(P)$  and  $f(Q)$  lie on the same side of  $f(M)$  in  $D'_2$ ;
3.  $P$  lies inside/on/outside  $C_1$  in  $D'_1$  iff  $f(P)$  lies inside/on/outside  $f(C_1)$  in  $D'_2$ ;
4.  $N$  intersects  $M$  at a point/along a segment in  $D'_1$  iff  $f(N)$  intersects  $f(M)$  at a point/along a segment in  $D'_2$ ;
5.  $M$  intersects  $C_1$  in  $D'_1$ , cutting it at one point/cutting it at two points/tangent at one point/tangent along a segment iff  $f(M)$  intersects  $f(C_1)$  in  $D'_2$ , cutting it at one point/cutting it at two points/tangent at one point/tangent along a segment; and
6.  $C_1$  has the same intersection signature with respect to  $C_2$  in  $D'_1$  that  $f(C_1)$  has with respect to  $f(C_2)$  in  $D'_2$ , where the intersection signature records if the two circles cross or are tangent at a point or along a segment each time they touch.

This is actually my restating of Mumma's definition, since that definition is divided into several pieces in different places. In clause 2 of the definition, when Mumma refers to a point  $P$  lying on a "given side" of a line  $AB$ , he appears to mean that  $P$  lies in the region either on left or right side of the line  $AB$  as we traverse the line from  $A$  to  $B$ . Thus, if  $P$  and  $Q$  lie on the left side of the line  $AB$  as we travel from  $A$  to  $B$  in  $D'_1$ , then this clause requires that  $f(P)$  and  $f(Q)$  lie on the left side of the line  $f(AB)$  as we travel from  $f(A)$  to  $f(B)$  in  $D'_2$ . Note that when we apply this clause, we may allow  $P$  and  $Q$  to be the same point.

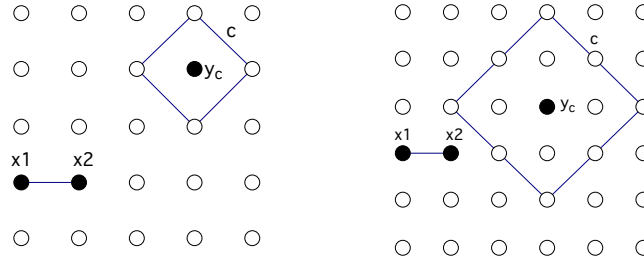
Mumma intends that two diagrams will be equivalent if they share the same *co-exact* features. The term *co-exact* was coined by Manders [4] to roughly mean those aspects of the diagram that don't change under small perturbations of the diagram, *i.e.*, its topological features. Mumma uses the term *co-exact* in a modified way that considers some of the topological features of the diagram, and also considers where the points in the diagram lie with



**Figure 2** A pair of diagrams that are equivalent under Definition 2.1.



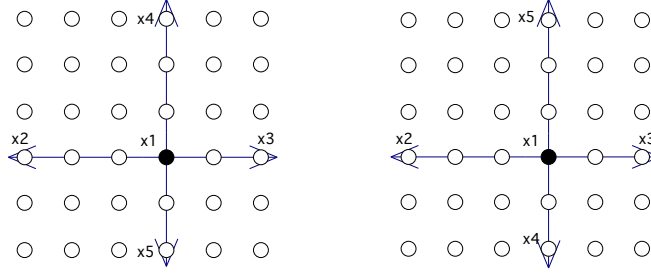
**Figure 3** A second pair of diagrams that are equivalent under Definition 2.1.



**Figure 4** A third pair of diagrams that are equivalent under Definition 2.1.

respect to the extensions of line segments and rays, even though this is a geometric rather than topological property. Mumma's intent is that the co-exact features of the diagrams will be exactly those features that should be information-bearing, and that two diagrams will be considered equivalent if they contain the same information.

However, Definition 2.1 does not quite function in the way Mumma intends. There are some features of the diagrams that appear to be information-bearing that aren't captured by this definition. Figure 2 shows a pair of diagrams that should have different co-exact features (under Mumma's use of the term co-exact) since  $x1$  occurs on different sides of the extension of the segment in the two diagrams. However, these diagrams are equivalent under Definition 2.1, since the definition doesn't say anything about the side of an extension of a segment that a point falls on. Likewise, Figure 3 shows a pair of diagrams that are topologically different but are equivalent under Definition 2.1, since the definition doesn't say anything about the order that points must occur in



**Figure 5** A fourth pair of diagrams that are equivalent under Definition 2.1.

along a circle. The diagrams in Figure 4 are equivalent because even though the circle crosses the extension of the line segment in the second diagram but not in the first, there are no points on the circle there, and when we complete a diagram, we only add points where objects in the diagram actually intersect, not where their extensions would intersect. (Note that, while there are eight dots on the circle, they are not filled in, and therefore aren't considered points in the diagram.) Finally, the diagrams in Figure 5 are equivalent because Definition 2.1 doesn't say anything about what happens to end arrows of rays and lines. In each of these pairs, we would expect that the two diagrams would convey different information, but in each case, Definition 2.1 classifies them as being equivalent.

**Eu** includes construction rules that are intended to formalize the kinds of ruler and compass constructions that are common in Euclidean proofs. (See Appendix A for a brief listing of **Eu**'s construction and proof rules.) However, because Definition 2.1 does not actually capture all of the information-bearing features of its diagrams, **Eu**'s construction rules are not well defined on equivalence classes of diagrams. In each of the pairs of diagrams presented in Figures 2 to 5, we can apply one of the construction rules to get diagrams that are not equivalent under Definition 2.1. In Figure 2, we can extend the segment  $(x_2, x_3)$  to a ray with endpoint  $x_2$ ; in one of the resulting diagrams,  $x_1$  lies on this ray, but not in the other. In Figure 3, we can connect  $x_2$  and  $x_4$  with a segment; in one of the resulting diagrams,  $x_1$  lies on the left of the segment from  $x_2$  to  $x_4$ , while in the other it lies on the right of this segment. In Figure 4, we can extend the line segment  $(x_1, x_2)$  to a ray with endpoint  $x_1$ ; in one diagram this ray will intersect the circle, and in the other it won't. Finally, in Figure 5, we can add a point to the ray  $(x_1, x_4)$ ; the new point will be on the left side of the line from  $x_2$  to  $x_3$  in one diagram, and on the right side in the other. Note that in **Eu**, the result of connecting points to form a segment is the actual straight line that results when the given points are connected, which is unique for a given diagram. Likewise, the result of extending a segment to a ray or a ray to a line is the actual straight line or ray through the given object.

**Eu** is set up so that proofs have two distinct stages: a construction stage followed by a demonstration stage. Each construction step that introduces a new geometric object into a proof must be performed twice: once in the construction stage using construction rules, and then a second time in the

demonstration stage, using positional rules. The positional rules used in the demonstration stage have much stricter restrictions placed on when they can be used than the construction rules, which can almost always be used. In three of the four examples just given, those for the diagrams in Figures 2 to 4, the constructions described can be carried out in the construction stage, but cannot be directly carried out during the demonstration stage, because of the additional restrictions. Nevertheless, being able to construct non-equivalent diagrams from equivalent diagrams even just at the construction stage is already quite problematic. It shows that we cannot treat diagrams as encoding the same information if they are equivalent, and makes it impossible to define a reasonable semantics for the diagrams under which all of **Eu**'s rules are sound.

In order to make sense of this claim, we now define a semantics for **Eu**.

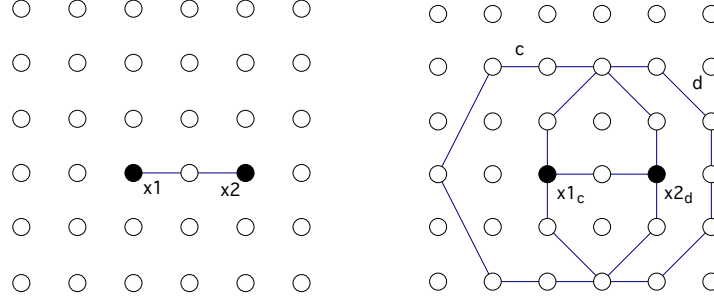
### 3 Semantics of **Eu**

Mumma does not explicitly define a formal semantics for his diagrams, but we can roughly define one possible semantics for the diagrams by saying that a collection  $M$  of points, lines, rays, line segments, and circles in the Euclidean plane should be a model of the diagram  $D$  if and only if  $D$  and  $M$  have the same co-exact features. More precisely, we can make the following definition:

**Definition 3.1** Let  $\Delta$  be a labeled diagram, let  $A$  be a metric assertion, and let  $M$  be a collection of points, lines, rays, line segments, and circles in the Euclidean plane. Furthermore, let  $I$  be the set of all intersection points of objects in  $M$  with one another, and let  $M'$  be the result of adding the points in  $I$  to  $M$ . Then  $M$  is a *model* of  $(\Delta, A)$  if and only if there exists a completion  $\Delta'$  of  $\Delta$  and a bijection  $f$  that matches the diagrammatic points, lines, rays, line segments, and circles in  $\Delta'$  with the points, lines, rays, line segments, and circles (respectively) in  $M'$ , such that  $A$  is true when interpreted in  $M'$  via  $f$  and the labeling, and such that for all diagrammatic points  $P$  and  $Q$ , diagrammatic linear elements (lines, rays, or segments)  $N_1$  and  $N_2$ , and circles  $C_1$  and  $C_2$  in  $\Delta'$ ,

1.  $P$  lies on  $N_1$  in  $\Delta'$  iff  $f(P)$  lies on  $f(N_1)$  in  $M'$ ;
2.  $P$  and  $Q$  lie on a given side of  $N_1$  in  $\Delta'$  iff  $f(P)$  and  $f(Q)$  lie on the same side of  $f(N_1)$  in  $M'$ ;
3.  $P$  lies inside/on/outside  $C_1$  in  $\Delta'$  iff  $f(P)$  lies inside/on/outside  $f(C_1)$  in  $M'$ ;
4.  $N_2$  intersects  $N_1$  at a point/along a segment in  $\Delta'$  iff  $f(N_2)$  intersects  $f(N_1)$  at a point/along a segment in  $M'$ ;
5.  $N_1$  intersects  $C_1$  in  $\Delta'$ , cutting it at one point/cutting it at two points/tangent at one point/ tangent along a segment iff  $f(N_1)$  intersects  $f(C_1)$  in  $M'$ , cutting it at one point/cutting it at two points/ tangent at one point/ tangent along a segment; and
6.  $C_1$  has the same intersection signature with respect to  $C_2$  in  $D'_1$  that  $f(C_1)$  has with respect to  $f(C_2)$  in  $M'$  (recall that the intersection signature records if the two circles cross or are tangent at a point or along a segment each time they touch).

Symbolically, if  $M$  is a model of  $(\Delta, A)$ , we write  $M \models (\Delta, A)$ .

Figure 6 Diagrams  $F_1$  and  $F_2$ .

Note that this definition is almost identical to the definition we gave for diagram equivalence. This semantics is a way of formalizing Manders's notion that the information-bearing pieces of a Euclidean diagram should be precisely its co-exact features [4].

With this definition, we can then define the notion of a geometric consequence analogously to the normal definition of a logical consequence as follows:

**Definition 3.2** Let  $\mathcal{D}$  be a set of labeled diagrams, let  $\mathcal{A}$  be set of metric assertions, let  $\Delta_2$  be a labeled diagram, and let  $A_2$  be a metric assertion. We define  $(\Delta_2, A_2)$  to be a **geometric consequence** of  $(\mathcal{D}, \mathcal{A})$  if and only if given any  $M_1$  that is a model of  $(\Delta_1, A_1)$  for every  $\Delta_1 \in \mathcal{D}$  and  $A_1 \in \mathcal{A}$ , we can extend  $M_1$  to a model  $M_2$  of  $(\Delta_2, A_2)$  by adding additional points, lines or circles to  $M_1$  if necessary. Symbolically, if  $(\Delta_2, A_2)$  is a geometric consequence of  $(\mathcal{D}, \mathcal{A})$ , we write  $(\mathcal{D}, \mathcal{A}) \vdash (\Delta_2, A_2)$ .

This definition of geometric consequence is an adaptation of that given by Luengo in [2]; it captures formally the idea of what it means for one diagram to follow from another. As an example, consider the diagrams  $F_1$  and  $F_2$  shown in Figure 6.  $F_1$  is the starting diagram for the proof of Euclid's Proposition 1 from Book I of the *Elements* [1]. It represents a single segment with endpoints  $x1$  and  $x2$ .  $F_2$  occurs in the third step of the proof of Proposition 1; it represents the same segment along with two circles. One of these circles is centered at  $x1$  and passes through  $x2$ , while the other is centered at  $x2$  and passes through  $x1$ . Since any model containing a single segment can be extended to a model in which this segment is a radius of two such circles,  $(\{F_1\}, \{\Theta\}) \vdash (F_2, \Theta)$ , where  $\Theta$  denotes the empty metric assertion (which is always true).

We can now define what it means for an inference rule in **Eu** to be sound:

**Definition 3.3** An inference rule  $R$  of **Eu** is **sound** if whenever  $(\Delta, A)$  can be obtained from the collections of diagrams and metric assertion sets  $\mathcal{D}$  and  $\mathcal{A}$  via the rule  $R$ , then  $(\mathcal{D}, \mathcal{A}) \vdash (\Delta, A)$ .

Sound rules are thus those that only allow you to deduce geometric consequences of their hypotheses.

For most formal systems, knowing whether or not its rules are sound is enough to determine whether or not all of its theorems are correct. However,

**Eu** is somewhat unusual in that a diagram may be derivable from another via some rule without any claim provable in **Eu** reflecting this fact. (*Claims* are the analogues of theorems in **Eu**; for a brief summary of **Eu**'s rules governing claims, see Appendix B.) For example, some diagram  $D_2$  may be obtained from  $D_1$  using construction rule  $R$ , but there may not be any positional rule that allows  $D_2$  to be derived from  $D_1$  in the demonstration stage. To account for this, we make the following definition:

**Definition 3.4** A provable atomic claim  $\vdash_{\mathbf{Eu}} \Delta, A$  is *correct* if and only if  $(\{\}, \{\}) \models (\Delta, A)$ , that is, if it is possible to extend any geometric arrangement of points, circles, and lines in the plane to a model of  $(\Delta, A)$ . A provable conditional  $\vdash_{\mathbf{Eu}} \Delta_1, A_1 \rightarrow \Delta_2, A_2$  is correct if and only if  $(\{\Delta_1\}, \{A_1\}) \models (\Delta_2, A_2)$ . The system **Eu** is correct if and only if every claim (atomic or non-atomic) that is provable in **Eu** is correct.

We would now like to define **Eu** to be inconsistent if it can derive a contradiction. In general, there are two ways that a contradiction could occur in **Eu**: two diagrams could contain contradictory information about the layout of some geometric objects, or two metric assertions could contradict one another. Mumma introduces the symbol  $\perp$  to denote a contradiction in metric assertions. He also defines two diagrams to be inconsistent if they each contain a subdiagram labeled by the same set of labels  $\vec{x}$ , but these two subdiagrams are not equivalent. Note that if two diagrams are inconsistent, then if we try to combine them using rule C9 or P19, then there will be no possible diagram that results. Mumma does not specify what should happen if we have diagrams that satisfy the hypotheses of an inference rule, but none that satisfy the conclusion. One possible solution would be to adopt the rule that if the hypotheses of some rule of inference are met, but there is no possible diagram representing the conclusion of the rule, then instead we infer  $\perp$ .

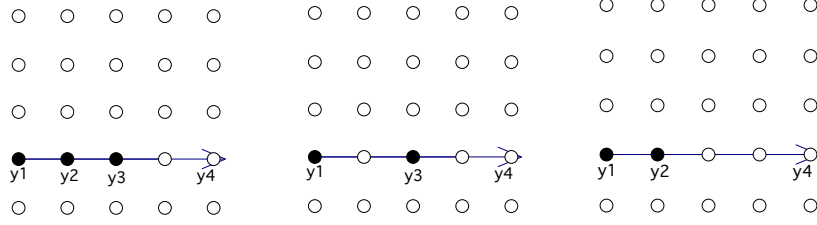
However, we do not need this rule to derive  $\perp$  from inconsistent diagrams. We can derive  $\perp$  from two inconsistent diagrams  $D_1$  and  $D_2$  in **Eu** as follows: derive diagram  $B$ , the first diagram shown in Figure 7. From diagram  $B$ , derive the two subdiagrams  $B_1$  and  $B_2$ , also shown in Figure 7. Combine  $B_1$  and  $B_2$ , using rule P19. This gives us the three cases  $C_1$ ,  $C_2$ , and  $C_3$  shown in Figure 8. Using the two inconsistent diagrams  $D_1$  and  $D_2$ , eliminate cases  $C_1$  and  $C_2$ , leaving case  $C_3$ . From a subdiagram of diagram  $B$  and rule Q8, derive the metric assertion  $y_1y_2 < y_1y_3$ , and from a subdiagram of  $C_3$  and rule Q8, derive  $y_1y_3 < y_1y_2$ . Using Q9 (Transitivity), derive  $y_1y_2 < y_1y_2$ . Finally, using Q10, derive  $\perp$ .

The derivation just given shows that we can derive  $\perp$  from either kind of contradiction. We therefore now define what it means for a system like **Eu** to be inconsistent as follows:

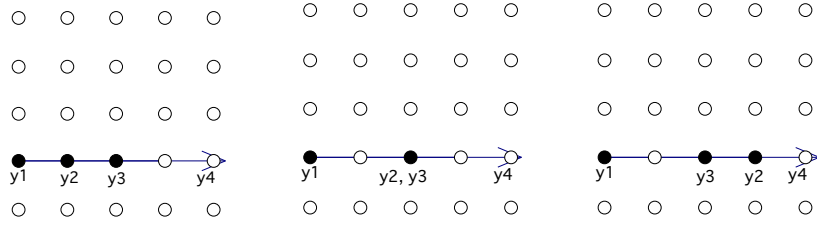
**Definition 3.5** A formal system like **Eu** is *inconsistent* if there exists a diagram  $D$  such that  $\vdash_{\mathbf{Eu}} D, \perp$ .

A related notion is that of completeness. A system like **Eu** is complete if every correct claim is provable in the system. However, it is well known that any system based on ruler and compass constructions is incomplete in this sense, because all ruler and compass constructions can be carried out in  $F \times F$  for





**Figure 7** Diagram  $B$  and its subdiagrams  $B_1$  and  $B_2$ .



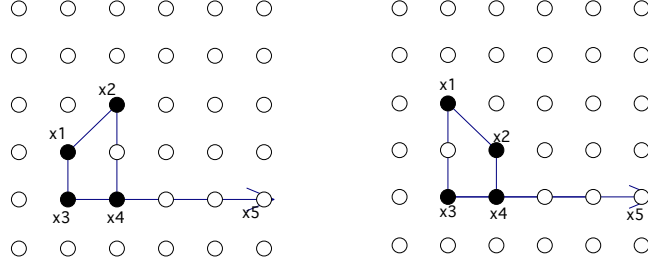
**Figure 8** Three cases  $C_1, C_2$ , and  $C_3$  obtained by combining diagrams  $B_1$  and  $B_2$ .

any Euclidean field  $F$ . (A Euclidean field is an ordered field that is closed under the extraction of square roots.) Any point that can be constructed with a ruler and compass must exist in every such model. Thus, no system based on ruler and compass constructions can prove the existence of objects that don't exist over all such fields, even if they do exist over the reals. One example of this phenomenon is the following: it is possible to write down a claim of **Eu** that expresses that any angle can be trisected. This is certainly a correct claim over the reals, but, as is well known, it is not true in all Euclidean fields and therefore cannot be proven in **Eu**.

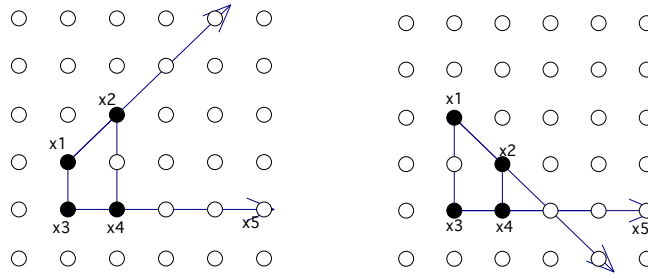
So the most we could hope for in the way of completeness is that any claim that is correct with respect to all Euclidean fields is provable in the system. (We can extend the notion of  $M$  being a model of  $D$  to any Euclidean field  $F$  in the natural way by allowing  $M$  to be a collection of points, lines, rays, segments, and circles in  $F \times F$  rather than in the Euclidean plane  $R \times R$ .) If any claim that is correct with respect to all Euclidean fields is provable in the system, then we say that the system is *complete with respect to Euclidean fields*.

#### 4 **Eu** is unsound, incorrect, and inconsistent

With these definitions in hand, we can immediately see that each of the examples from Figures 2 to 5 can be used to show that a rule of **Eu** is unsound. For example, in Figure 2, the right hand diagram is a model of the left hand diagram, but if we extend the segment  $(x_2, x_3)$  to a ray with endpoint  $x_2$  in the left hand diagram, there won't be a way to extend the right hand diagram so that it is still a model, because if we extend the segment to a ray



**Figure 9** Equivalent diagrams  $D_1$  and  $D_2$ .

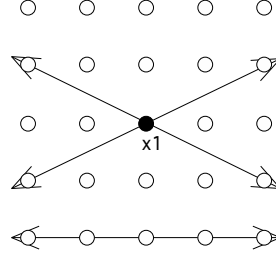


**Figure 10** Non-equivalent diagrams  $D'_1$  and  $D'_2$ , obtained by extending the segment through  $x_1$  and  $x_2$  to a ray in  $D_1$  and  $D_2$ .

in the model,  $x_1$  will lie on the ray. A similar argument applies to the other examples.

This shows that several of the rules of **Eu** are unsound. However, the real problem in these examples is that the definition of equivalent diagrams doesn't take account of all of the co-exact features of a diagram. Thus, we might hope that we could make **Eu** sound by fixing Definition 2.1. Unfortunately, this is not the case. Consider the two diagrams  $D_1$  and  $D_2$  shown in Figure 9. These two diagrams are equivalent according to Definition 2.1, and correctly so: they really do share all of the same co-exact features. They are topologically equivalent, and every point falls on the same side of each segment in both diagrams. However, if we extend the segment  $(x_1, x_2)$  to a ray with endpoint  $x_1$  in each, using rule C5 or P13, we get the two non-equivalent diagrams  $D'_1$  and  $D'_2$  shown in Figure 10. Since diagram  $D_1$  is a model of diagram  $D_2$  that cannot be extended to a model of  $D'_2$ , rules C5 and P13 are unsound.

Similar arguments can be made to show that many of the rules of **Eu** are unsound. Unsound rules of **Eu** include C3, C4, C5, C6, C7, C9, P3, P5, P9, P10, P11, P12, P13, P16, and P17. Most of these rules are unsound because we can find equivalent arrangements of points, lines, and circles that result in non-equivalent arrangements after we perform the indicated operation. P12 is unsound because the inequality in the hypothesis is backwards, and P16 and P17 are unsound because their conclusions are only true if the radius of circle  $d$  is shorter than the radius of circle  $c$ , which isn't true in all models of the premises. Q7, Q8, and Q16 are also unsound as stated by Mumma; each



**Figure 11** Diagram  $D_3$ . This diagram contains no circles but is unsatisfiable.

one contains typographical errors that make it unsound. For details, see the discussion of these rules in Appendix A.

Now consider the following derivation in **Eu**. Start with diagram  $D_1$ . Using the diagram substitution rules C2 and P2, which allow us to substitute equivalent diagrams, derive diagram  $D_2$ . Apply rules C5 and P13 to derive diagrams  $D'_1$  and  $D'_2$ . Using the derivation given in the previous section, from the inconsistent diagrams  $D'_1$  and  $D'_2$  derive  $\perp$ .

This shows  $\vdash_{\mathbf{Eu}} D_1, \Theta \rightarrow D_1, \perp$ , where  $\Theta$  denotes the empty metric assertion. Since  $D_1, \Theta$  has a model, but there is no possible model of  $D_1, \perp$ , this claim is incorrect, and therefore **Eu** is incorrect.

Finally, note that  $\vdash_{\mathbf{Eu}} D, \Theta$  automatically for any diagram  $D$  not containing any circles, by the Diagrams without circles rule given in Appendix B. Since  $D_1$  does not contain any circles, this means  $\vdash_{\mathbf{Eu}} D_1, \Theta$ . Since  $\vdash_{\mathbf{Eu}} D_1, \Theta$  and  $\vdash_{\mathbf{Eu}} D_1, \Theta \rightarrow D_1, \perp$ , by **Eu**'s Modus Ponens rule we obtain  $\vdash_{\mathbf{Eu}} D_1, \perp$ . Thus, **Eu** is inconsistent.

As an aside, we note that the rule that allows us to derive any diagram not containing any circles is also unsound with respect to our semantics. To see this, consider diagram  $D_3$  shown in Figure 11. This diagram does not contain any circles, so  $\vdash_{\mathbf{Eu}} D_3, \Theta$ . However, this diagram is unsatisfiable. Mumma's definition of the completion of a diagram adds points wherever elements of the diagram intersect within the confines of the array of dots it is embedded in, even if the intersection does not occur at a dot. However, it does not add points to represent intersections if these intersections would occur outside the array of dots. Thus, the diagram shown in Figure 11 is its own completion. Therefore, according to our semantics, it must represent two infinite lines that do not intersect a third line but do intersect each other. (Note that if two infinite lines in a model of a diagram intersect anywhere, then their intersection point gets included and must be mapped to a point in the completion of the diagram.) Since it is impossible to find two lines that do not intersect a third line but do intersect each other in the Euclidean plane, this is another example of an incorrect claim provable in **Eu**.

Furthermore, **Eu** is able to prove  $\vdash_{\mathbf{Eu}} D_3, \Theta \rightarrow D_3, \perp$ , following the proof of Euclid's Proposition 30 of Book I of the *Elements* [1]. The proof is roughly

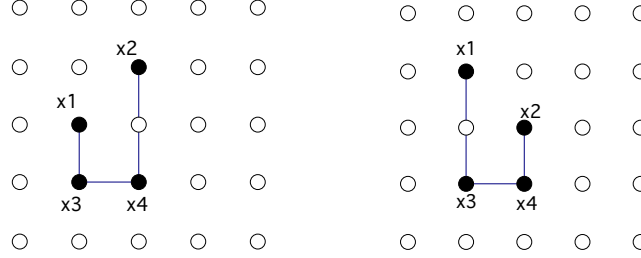
as follows: **Eu** can prove Euclid’s Proposition 29, which says that when non-intersecting (*i.e.*, parallel) infinite lines are cut by a transversal, the corresponding angles must be equal. **Eu** can prove this proposition as Euclid does, using rules P20 (Angle Trichotomy) and P12 (Parallel Postulate) to show that if the corresponding angles aren’t equal, then the lines must intersect by the parallel postulate. Thus, if  $L_1$  and  $L_2$  are both parallel (non-intersecting) with  $L_3$ , but intersect each other at point  $x_1$ , then we can draw a transversal  $T$  through point  $x_1$  and through  $L_3$ . The angles  $\alpha_1$  and  $\alpha_2$  that  $L_1$  and  $L_2$  make with the transversal must both be equal to the corresponding angle that  $T$  makes with  $L_3$ , so they must be equal to one another, and we can derive the metric assertion  $\alpha_1 = \alpha_2$ . However, one of these angles (without loss of generality, assume that it is  $\alpha_1$ ) is contained within the other. So by Q8, we can also derive the metric assertion  $\alpha_1 < \alpha_2$ . By substitution (Q6), this gives us the metric assertion  $\alpha_2 < \alpha_1$ , and so by Q10 we can derive  $\perp$ . Thus,  $\vdash_{\mathbf{Eu}} D_3, \perp$ , giving us another example of **Eu**’s inconsistency.

We could, of course, change our semantics to make diagram  $D_3$  satisfiable, but the fact that **Eu** can prove  $D_3, \Theta \rightarrow D_3, \perp$  suggests that our semantics is correct, and  $D_3$  should not be satisfiable; rather, the rule that allows us to derive any diagram not containing circles should be changed.

## 5 Can Eu be fixed by changing the definition of equivalent diagrams?

We saw that some of the problems with **Eu** could be fixed by fixing Definition 2.1. A natural question then, given the example of diagrams  $D_1$  and  $D_2$  from Figure 9, is: can the rules and definitions of **Eu** be further modified to make the system sound and consistent? Unfortunately, it appears that they cannot.

At first glance, it might appear that we could fix the problem by further modifying Definition 2.1 so that diagrams  $D_1$  and  $D_2$  are no longer equivalent. These diagrams are clearly topologically equivalent, so any change to Definition 2.1 that makes them not equivalent will have moved us further from the idea that the co-exact features of a diagram are those that are topological. Nevertheless, there is an change that we can try. The problem with these diagrams arose from the fact that when we extended a segment, the extension of the segment intersected an existing line differently. Recall that we defined two diagrams to be equivalent if their completions had certain properties in common, where a completion of a diagram  $D$  is a new diagram  $D'$  that contains the same geometric elements but such that the number of dots in the underlying square array of dots is increased so that every intersection point between two geometric elements occurs at a dot in the array and is added as a point to the diagram. We could redefine the completion of a diagram to also include as a point every intersection point of the extension of any geometric object in the diagram with any other geometric object or extension of a geometric object. (This change was suggested to me by John Mumma in response to an earlier draft of this paper.) If this change was successful, it would suggest that we should consider the information-bearing (co-exact) features of a diagram to be its topological features along with information about where the objects in the diagram lie with respect to all possible extensions of

Figure 12 Diagrams  $D_1^*$  and  $D_2^*$ .

its segments and rays. Under this new definition, diagrams  $D_1$  and  $D_2$  would no longer be equivalent.

However, we would just have pushed the problem back one step. Consider diagrams  $D_1^*$  and  $D_2^*$ , shown in Figure 12. These diagrams would be equivalent under our new definition, but when we connected  $x1$  and  $x2$  in these diagrams using rules C4 and P11, we would get non-equivalent diagrams that could be used to derive a contradiction as before.

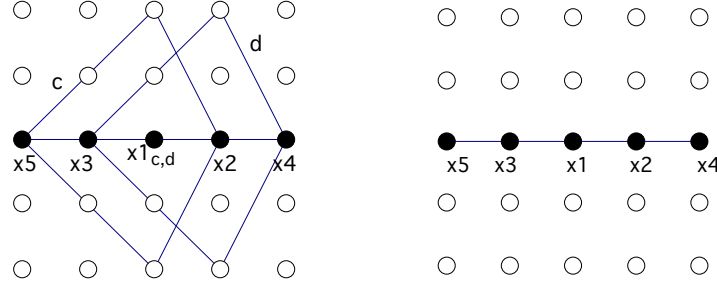
In fact, no matter what changes we make to the definition of equivalent diagrams, as long as there is any way to perform a construction that would lead to more than one possible non-equivalent resulting case, we will have the same problem. Thus, it seems that it will not be possible to make **Eu** sound merely by changing the definition of equivalent diagrams. The problem is deeper, and seems to stem from the fact that **Eu** doesn't have a way to branch into cases.

## 6 Restriction of Positional Rules

Another possible way to try to make **Eu** correct and consistent would be to try to further restrict the positional rules. In general, **Eu** is designed so that there are very few restrictions on when the construction rules can be applied, but there are many restrictions on how the corresponding positional rules can be applied. Any diagram that is part of a claim of **Eu** must in effect be derived twice, once in the construction stage using construction rules, and once in the demonstration stage, using positional rules. Mumma's intention behind this setup is that a complete diagram will be able to be derived in the construction stage to show one example of how the construction works, and then general facts true in any situation can be derived in generality in the demonstration stage.

We cannot significantly restrict the construction rules of **Eu** in order to try to regain consistency, because if we do, we will no longer be able to use **Eu** to try to duplicate Euclid's complete constructions. However, another approach that we might try would be to further restrict the positional rules. Since in most cases, constructions must be done in both stages, if we restrict what can be done in the demonstration stage, we will be able to eliminate some of the inconsistencies that can be derived.

Unfortunately, this approach also suffers from a number of significant problems. One is that as long as there are still unsound rules in the system, even if



**Figure 13** Diagrams  $E_1$  and  $E_2$ .

they are only in the construction stage, we may still be able to use them to derive a contradiction. Another is that the restrictions placed on the positional rules have weakened the system so that there are already many simple correct geometric consequences that the system cannot derive, and further weakening the rules will exacerbate this problem. We will address these issues in turn.

First, we will show how a contradiction can be derived in **Eu** using unsound construction rules, without using any positional rules at all. Consider diagram  $E_1$ , the first diagram shown in Figure 13. This diagram shows two circles with a common center point which intersect but do not coincide. **Eu** is able to derive the (correct) claim  $\vdash_{\mathbf{Eu}} E_1, \Theta \rightarrow E_1, \perp$  using rules Q15, Q9, and Q10. Now, consider diagram  $E_2$ , the second diagram in Figure 13. Starting with diagram  $E_2$ , we can derive diagram  $E_1$  in the construction stage by two applications of rule C7. Next, using modus ponens and the previously proven claim  $\vdash_{\mathbf{Eu}} E_1, \Theta \rightarrow E_1, \perp$ , we can end the construction stage with a context that includes diagram  $E_1$  and the metric assertion  $\perp$ . In the demonstration stage, we start with diagram  $E_2$  and use rule Q1 (Metric inference from the context) to derive  $\perp$ . We finish the proof and conclude  $\vdash_{\mathbf{Eu}} E_2, \Theta \rightarrow E_2, \perp$ . Notice that since  $E_2$  doesn't contain any circles,  $\vdash_{\mathbf{Eu}} E_2, \Theta$ . Thus, as before, by modus ponens, we obtain  $\vdash_{\mathbf{Eu}} E_2, \perp$ . So we can derive an inconsistency in **Eu** from an unsound construction rule (in this case, C7) without using any unsound positional rules.

It may of course be possible to modify the rules of the system so that no information from the construction stage can be used in the demonstration stage, so that the unsoundness of the construction rules cannot be used to derive an incorrect conclusion. If the construction rules are unsound and cannot be used in the derivation of any conclusion, however, then they aren't really a meaningful part of the system.

The other reason that trying to place further restrictions on the inference rules is problematic is that these rules weaken the system, and there are already many elementary geometric facts that the system is too weak to prove. For example, we can prove the following theorem:

**Theorem 6.1** *Let  $D_1$  and  $D_2$  be two diagrams such that  $D_1$  contains a circle  $c_1$  and  $D_2$  contains  $c_1$  along with a second circle  $c_2$  not in  $D_1$ . Also, let  $A_1$  and  $A_2$  be metric assertions. Then **Eu** cannot prove the claim  $D_1, A_1 \rightarrow D_2, A_2$ .*

**Proof:** The diagram combination rule P19 can only be applied to two diagrams if one of them only has a single element not occurring in the other diagram and that element is a point, ray, or infinite line. Therefore, the diagram combination rule can never be used to combine two diagrams if one contains a circle and the other contains a different circle. Furthermore, no other rules allow you to add a circle to a diagram already containing a circle. The only rule that could potentially allow you to add such a circle would be P18, Modus Ponens, but this would require us to have already proven a claim of the given form. Since no claim of this form can be proven without having already proven a claim of this form, we see that no claim of this form can be proven at all.  $\square$

**Corollary 6.2** *Let  $D_1$  and  $D_2$  be two diagrams such that  $D_2$  contains two circles not in  $D_1$ , and let  $A_1$  and  $A_2$  be metric assertions. Then **Eu** cannot prove the claim  $D_1, A_1 \rightarrow D_2, A_2$ .*

As an illustration of the effect of these theorems, let us recall the diagrams  $F_1$  and  $F_2$  from Figure 6. As previously discussed,  $F_2$  is a geometric consequence of  $F_1$ , but since  $F_2$  contains two circles not in  $F_1$ , **Eu** cannot prove  $F_1, \Theta \rightarrow F_2, \Theta$ .

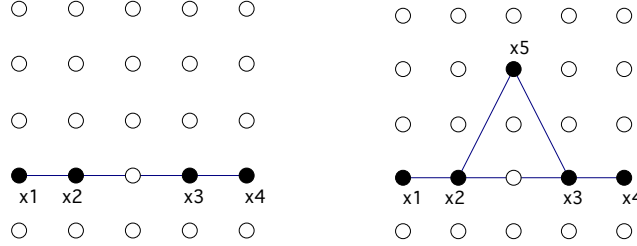
This answers a question posed but left open in Mumma's article [10, Footnote 9] and dissertation [7, Section 1.6.1, Footnote 14]: is **Eu** complete with respect to Euclidean fields? Here, we see that it is not, since  $F_1, \Theta \rightarrow F_2, \Theta$  is a correct claim relative to any Euclidean field that is expressible in **Eu** but not provable in **Eu**.

Similarly, we can prove the following theorem:

**Theorem 6.3** *If  $\vdash_{\mathbf{Eu}} D_1, A_1 \rightarrow D_2, A_2$ , then  $D_2$  cannot contain a circle along with two linear elements  $L_1$  and  $L_2$  that are not collinear with the center of the circle unless the circle and linear elements collinear with  $L_1$  and  $L_2$  are already present in  $D_1$ .*

**Proof:** We can easily verify this by checking that each of the inference rules from the demonstration stage preserves this property:

1. Rules P1–P3 and P13 don't introduce any new circles or linear elements that aren't collinear with already existing linear elements;
2. the conclusions of rules P4–P8, P11, and P12 can't contain any circles;
3. the conclusions of rules P9, P14, P16, and P17 each contain a single circle and don't contain any linear elements not collinear with the center of the circle;
4. the conclusions of rules P10 and P15 each contain a single circle and contain at most one linear element not collinear with the center of the circle;
5. for rule P18, if we assume, inductively, that in the implication  $\Delta, A \rightarrow \Gamma, B$ , the diagram  $\Gamma$  doesn't contain a circle and two linear elements not collinear with the center of the circle unless  $\Delta$  does, then the conclusion of this rule likewise won't contain a circle and two linear elements not collinear with the center of the circle unless  $\Delta$  does in the premise of the rule;



**Figure 14** Diagrams  $G_1$  and  $G_2$ .

6. P19 cannot be used to add a linear element to a diagram containing a circle or used to add a circle to any diagram, so if the conclusion contains a circle and any linear elements, then the circle and linear elements must have already been in both diagrams P19 was applied to; and
7. P20 and Q1–Q16 are used to derive metric assertions, not diagrams.

□

This result shows that almost none of the propositions in Book IV of Euclid's *Elements* are provable in **Eu**, since these mainly show how to inscribe or circumscribe a polygon in or around a circle or a circle in or around a polygon.

Likewise, we can show the following:

**Theorem 6.4** *If  $\vdash_{\mathbf{Eu}} D_1, A_1 \rightarrow D_2, A_2$ , then  $D_2$  cannot contain two non-intersecting lines  $L_1$  and  $L_2$  unless  $L_1$  and  $L_2$  are both already present in  $D_1$ .*

**Proof:** As above, we can verify this by checking that each inference rule preserves this property. The only rules that could potentially allow us to add a line  $L_1$  to an existing diagram containing line  $L_2$  are P13 and P19. However, neither rule will apply here, since  $L_2$  doesn't intersect  $L_1$  and it cannot form part of a convex broken line that contains or intersects  $L_1$ . □

This result shows that **Eu** cannot prove what is a key proposition of Book I of the *Elements*, Proposition 31, which shows how to construct a line parallel to a given line through a given point. In fact, we see from the preceding theorems that there are several large classes of correct claims that are expressible but unprovable in **Eu**.

We should also note that because **Eu** does not allow case branching within proofs, many proofs that rely on this case branching will not be expressible in **Eu**. For example, consider Proposition 22 of Book I of the *Elements*, which shows how to construct a triangle given three side lengths satisfying the triangle inequality. Let  $G_1$  and  $G_2$  be the diagrams shown in Figure 14, let  $A$  be the metric assertion

$$x_1x_2 + x_2x_3 > x_3x_4 \ \& \ x_1x_2 + x_3x_4 > x_2x_3 \ \& \ x_2x_3 + x_3x_4 > x_1x_2,$$

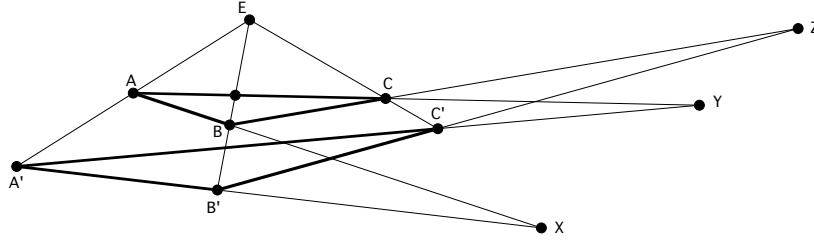
and let  $B$  be the metric assertion

$$A \ \& \ x_1x_2 = x_2x_5 \ \& \ x_3x_4 = x_3x_5.$$



Then one version of Proposition 22 is given by the claim  $G_1, A \rightarrow G_2, B$ . If **Eu** allowed case branching, we could give a proof of this claim as follows: first, draw a circle around point  $x_2$  with radius  $x_1x_2$ , and likewise, draw a circle around point  $x_2$  with radius  $x_3x_4$ . These circles may or may not intersect, and can intersect the line  $x_1x_4$  in a variety of ways; for example, the first circle might intersect the line between  $x_2$  and  $x_3$ , or at  $x_3$ , or between  $x_3$  and  $x_4$ , or at  $x_4$ , or past  $x_4$ . A formal system like **FG** that enumerates all of the topological possibilities that can result from applying a construction rule would tell us all of the cases we had to consider. In each case, one of two things will happen: either the two circles will intersect, in which case we can connect  $x_2$  and  $x_3$  to one of the intersection points to obtain  $G_2$ ; or else one of the inequalities in  $A$  will be violated, so we will be able to eliminate the diagram. So diagram  $G_2$  will be a subdiagram of all of the remaining diagrams and we will be able to draw the desired conclusion. This is more or less the argument that Euclid gives, although, as usual, he gives the argument for a single case and leaves it to the reader to fill in the other cases. However, without allowing case branching, it doesn't appear that there is any way to represent this proof in **Eu**.

In this case, Euclid's proposition was expressible, but not apparently provable in **Eu**. However, there are cases in which a proposition isn't even expressible in **Eu** because of the extra information contained in **Eu**'s diagrams about which side of a line segment each point lies on. For example, consider the Pythagorean Theorem, Euclid's Proposition 47 of Book I. In the proof of this theorem, Euclid starts with a right triangle, constructs squares on each of the sides, and then proceeds to show that the area of the square on the hypotenuse is equal to the sum of the areas of the other two squares. Mumma discusses this theorem in Section 1.6 of his thesis, where he identifies this theorem with a claim that starts with a diagram containing a single right triangle and concludes with a diagram in which the triangle has squares drawn on the sides, along with the metric assertion that the area of the square on the hypotenuse is equal to the sum of the areas of the other two squares. Let's call this Claim C. Mumma correctly points out that this concluding diagram fixes additional information in his system, such as whether or not the corner of the square on the hypotenuse lies above or below or on the line created by the side of one of the other squares. Mumma then asserts that we have to prove an additional conditional for each such possibility. Actually, however, the situation is worse: we won't be able to prove Claim C in any of the cases, because we have a single hypothesis diagram that branches into several cases during the course of the construction, and **Eu** doesn't allow case branching. In fact, Claim C isn't even a correct claim in **Eu**, because in addition to the usual content of the Pythagorean Theorem, it also claims the additional information about where the corners of the squares lie with respect to the sides of the other squares. It may be possible to start with a hypothesis diagram in which the squares are already drawn in a particular case, and prove the Pythagorean Theorem in that case. The resulting claim won't have the same content as Euclid's version of the Pythagorean Theorem, though, since Euclid's proposition is a claim about all right triangles, not just right triangles with squares already constructed on the sides.

Figure 15 Diagram  $H$ 

To see why this makes a difference, consider Euclid's proof of the next proposition, Proposition 48, which shows that if the sum of the squares on two sides of a triangle is equal to the square on the third side, then the triangle is right. During the course of this proof, Euclid constructs a right triangle and then applies Proposition 47, the Pythagorean Theorem, to it. If we try to duplicate this proof in **Eu**, we will be able to construct the right triangle, but then we will not be able to continue the proof, because we won't be able to apply the Pythagorean Theorem to a triangle that doesn't already have squares drawn on the sides. Nor will we be able to add the squares to the sides in **Eu**, since as we have already noted, there are multiple non-equivalent cases that could occur when we try to add the squares. Thus, it does not appear that there will be any way to duplicate this proof in **Eu**.

## 7 Combination Rules in **Eu**

There is one additional major problem with **Eu**'s rules that stems from the fact that straight lines are represented by genuinely straight lines. This problem arises in the combination rules.

We have already mentioned one problem with the combination rules: namely, it isn't clear how they are supposed to work when there is no diagram that satisfies the conclusion of the rules, even though we have diagrams that satisfy the premises. This could be because we have two inconsistent diagrams, but with rule C9 this can happen even when the diagrams are consistent. This is because rule C9 doesn't allow a line from one diagram to be coincident with a point from the other diagram, but they may coincide in every possible combination of the two diagrams. For example, consider diagram  $H$  shown in Figure 15. This diagram shows two triangles  $ABC$  and  $A'B'C'$  such that the lines connecting the corresponding vertices of the two triangles are concurrent and intersect at point  $E$ . (The two triangles have been drawn with bold lines in order to make them easier to see.) Points  $X$ ,  $Y$ , and  $Z$  mark the intersection points of the corresponding sides of these triangles. According to Desargues' Theorem, points  $X$ ,  $Y$ , and  $Z$  must be collinear in this diagram and in any diagram equivalent to it. Now, consider a diagram  $H'$  obtained by taking the subdiagram of  $H$  that just contains  $X$  and  $Y$ , adding the segment  $XY$ , and extending it to a line  $L$ . If we try to combine diagrams  $H$  and  $H'$  using rule C9, there will be no possible

combination that fits the restriction that line  $L$  can't pass through point  $Z$ , even though  $H$  and  $H'$  are consistent.

Furthermore, if we weren't aware of Desargues' Theorem, we would have a very hard time recognizing that there wasn't any possible diagram that could serve as the combination of  $H$  and  $H'$ . Since there are an infinite number of diagrams that are equivalent to  $H$ , we could go on forever checking to see if the line through  $x$  and  $y$  in each diagram goes through  $z$ . This example illustrates the fact that there isn't an obvious simple algorithm for determining if there is a diagram that can result from applying rule C9.

One way we might attempt to create an algorithm to solve this problem would be to determine all of the potential topological arrangements that could conceivably result when the diagrams are combined (using an algorithm like that of **FG**), and then to try to determine if any of these arrangements can actually be satisfied by straight lines. In general, it is shown in [5] that deciding if a diagram of **FG** is satisfiable by an arrangement of actual straight lines, circles, and points is a decidable but NP-hard problem, so it can't always be decided in a tractable amount of time. However, this decision procedure works by checking if a certain first order sentence is true over the real numbers. **Eu** has the additional requirement that the points must be at integer coordinates. This is a problem, since the theory of real arithmetic is decidable, but the theory of integer arithmetic is not. So this problem may not be decidable at all. Given a potential arrangement of lines we should be able to write a system of Diophantine (integer-valued) equations and inequalities that has a solution if and only if the arrangement is realizable. It isn't clear if the question of whether or not this system has a solution will be decidable, but if it is, it will almost certainly be an NP-hard problem, since the simpler problem of solving a system of linear Diophantine equations is known to be NP-hard.

So it isn't clear if it is decidable if there is a possible diagram that can result when rule C9 is applied to two given diagrams, but if it is, then it will still almost certainly sometimes take an intractable amount of time to decide this.

Now, consider the diagram combination rule P19 from the demonstration stage. We might expect that P19 would suffer from the same problems as C9. However, because the set of diagrams that P19 can be applied to is severely limited, it appears that we probably will be able to determine all of the possible diagrams that could result in this case, although Mumma has not given an explicit algorithm for how to do so. Even so, this example shows that the restrictions governing the diagrams to which rule P19 applies cannot be relaxed very far. If we were allowed to apply P19 to diagrams  $H$  and  $H'$ , then when we tried to determine what diagrams could result, we would run into problems similar to those we ran into with rule C9. This is unfortunate, because if we could find a way to make rule P19 apply to arbitrary diagrams, then it wouldn't matter that we had limited the application of all of the other rules, since we could always add new circles or lines to a given diagram by adding them in a separate diagram, and then combining the two diagrams. As it stands, without relaxing the requirement that lines be represented by straight lines and instituting a more robust system of case analysis, **Eu** will

necessarily be quite limited in what new geometric objects can be added to a given diagram and therefore in what the system can prove.

## 8 Conclusion

Mumma claims in [7], [10], [8], and [9] that **Eu** is a better diagrammatic formalization of Euclidean geometry than **FG** because it avoids the case analysis that is a central part of **FG**'s methodology. However, we have shown here that **Eu** is inconsistent, unable to derive many simple correct claims, and contains a construction rule that may be undecidable. These problems are due, in part, to **Eu**'s lack of case analysis. It may be possible to fix **Eu** in order to make it consistent by changing the definition of equivalent diagrams and further restricting its positional rules (although the system with these changes will still contain unsound construction rules). However, making these changes will further weaken the system, which already cannot derive many elementary correct claims expressible in the system, and, as we have shown, cannot derive several propositions of Euclid's Book I of the *Elements*.

In order to be considered a good model for the way diagrams are used in geometry, we would want a diagrammatic formal system, at a minimum, to be

1. correct,
2. consistent, and
3. strong enough to duplicate most elementary geometric arguments of the kind found in the first four books of the *Elements*.

**Eu** violates all of these conditions, and it does not appear likely that there will be any way to modify the system to meet them without relaxing the requirement that diagrammatic lines be actually straight and adopting a system of case analysis similar to that of **FG**.

## Appendix A Construction and Proof rules of Eu

A brief summary of the Construction and Proof rules of **Eu** are given here for reference. The rules given here have been restated in a slightly different form than in the original and some details have been glossed over; the reader is referred to Mumma's Thesis [7] for complete details and much more explanation.

### A.1 Construction Rules

- C1: Diagram Extraction:** From  $\Delta$ , deduce  $\Gamma$ , where  $\Gamma$  is a sub-diagram of  $\Delta$ .
- C2: Diagram Substitution:** From  $\Delta$ , deduce  $\Gamma$ , where  $\Gamma$  is equivalent to  $\Delta$ .
- C3: Adding a Point:** From  $\Delta$ , deduce  $\Gamma$ , where  $\Gamma$  is the result of adding a new point to  $\Delta$ , where the new point cannot be at a dot in the array that is already collinear with three other elements in the diagram.
- C4: Adding a Segment:** From  $\Delta$ , deduce  $\Gamma$ , where  $\Gamma$  is the result of adding to  $\Delta$  a segment that connects two previously existing points.
- C5: Extending a segment to a ray:** From  $\Delta$ , deduce  $\Gamma$ , where  $\Gamma$  is the result of adding to  $\Delta$  a ray extending an existing segment.

- C6: Extending a ray to a line:** From  $\Delta$ , deduce  $\Gamma$ , where  $\Gamma$  is the result of adding to  $\Delta$  a line extending an existing ray.
- C7: Adding a circle with a given radius:** From  $\Delta$ , deduce  $\Gamma$ , where  $\Gamma$  is the result of adding to  $\Delta$  a diagrammatic circle with the given radius; the diagrammatic circle must be a convex polygon containing the given radius.
- C8: Modus Ponens:** From  $\Delta, A$  and  $\Delta, A \rightarrow \Gamma, B$ , deduce  $\Gamma, B$ .
- C9: Diagram Combination:** From  $\Delta_1$  and  $\Delta_2$ , deduce  $\Gamma$ , where  $\Gamma$  is the combination of  $\Delta_1$  and  $\Delta_2$ ,  $\Gamma$  is consistent with every previously derived diagram in the context, and any element in  $\Delta_1$  but not  $\Delta_2$  does not coincide in  $\Gamma$  with any element in  $\Delta_2$  but not  $\Delta_1$ .
- C10: Empty Assertion Introduction:** The empty metric assertion  $\Theta$  can be introduced at any point in a derivation without any premises.
- C11: Metric Assertion Conjugation:** From metric assertions  $A$  and  $A'$  deduce  $A \& A'$ .
- C12: Atomic Metric Assertion Extraction:** From the metric assertion  $A$ , where  $A$  is a conjunction of atomic metric assertions, deduce  $A'$ , where  $A'$  is one of the conjuncts.

## A.2 Positional Rules

- P1: Diagram Extraction:** From  $\Delta$ , deduce  $\Gamma$ , where  $\Gamma$  is a sub-diagram of  $\Delta$ .
- P2: Diagram Substitution:** From  $\Delta$ , deduce  $\Gamma$ , where  $\Gamma$  is equivalent to  $\Delta$ .
- P3: Point Introduction:** From  $\Delta$ , deduce  $\Gamma$ , where  $\Gamma$  is the result of adding a point labeled by  $x$  to  $\Delta$  such that all labels in  $\Delta$  precede  $x$  by  $\triangleright$  and no segment in  $\Delta$  is extended by the definition. See [7] for an explanation of what these conditions mean.



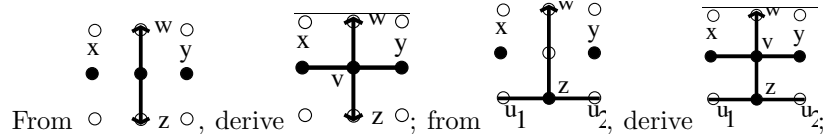
- P4: Segment Introduction:** From  $\circ \circ \circ$ , derive  $\circ \circ \circ$ .

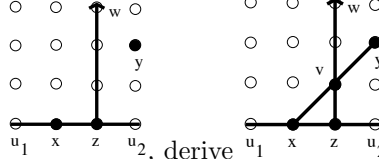
- P5: Segment Introduction with lines and rays:** From  $\delta[x, y, \vec{z}]$ ,



deduce  $\gamma[x, y, \vec{z}]$ , where  $\delta[x, y, \vec{z}]$  contains  $\circ \circ \circ$  as a sub-diagram, all other elements of  $\delta[x, y, \vec{z}]$  are linear elements incident with  $x$  and not with  $y$ , and  $\gamma[x, y, \vec{z}]$  is the result of adding the segment joining  $x$  and  $y$  in  $\delta[x, y, \vec{z}]$ .

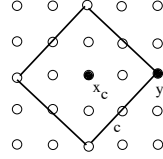
- P6–P8: Segment intersection with rays and lines:**





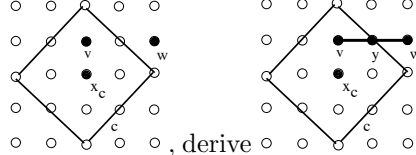
or from  $u_1$   $x$   $z$   $u_2$ , derive  $u_1$   $x$   $z$   $u_2$ ; where, in each case,  $u_1$  and  $u_2$  label either endpoints or end-arrows, and are possibly identical to  $x$  or  $z$ .

**P9: Segment Introduction with circle:** From  $\delta[x, y, \vec{z}]$ , deduce  $\gamma[x, y, \vec{z}]$ ,



where  $\delta[x, y, \vec{z}]$  contains  $\begin{array}{c} \circ \quad \circ \quad \circ \quad \circ \\ \circ \quad \circ \quad \circ \quad \circ \\ \circ \quad \circ \quad \circ \quad \circ \\ \circ \quad \circ \quad \circ \quad \circ \end{array}$  as a sub-diagram, all other elements of  $\delta[x, y, \vec{z}]$  are linear elements incident with  $x$  and not with  $y$ , and  $\gamma[x, y, \vec{z}]$  is the result of adding the segment joining  $x$  and  $y$  in  $\delta[x, y, \vec{z}]$ .

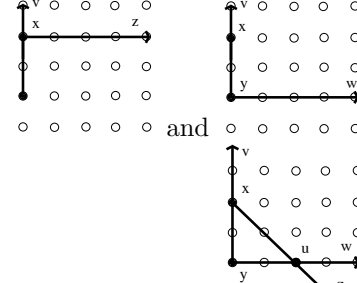
**P10: Segment Intersection with circle:**



From  $\begin{array}{c} \circ \quad \circ \quad \circ \quad \circ \\ \circ \quad \circ \quad \circ \quad \circ \\ \circ \quad \circ \quad \circ \quad \circ \\ \circ \quad \circ \quad \circ \quad \circ \end{array}$ , derive  $\begin{array}{c} \circ \quad \circ \quad \circ \quad \circ \\ \circ \quad \circ \quad \circ \quad \circ \\ \circ \quad \circ \quad \circ \quad \circ \\ \circ \quad \circ \quad \circ \quad \circ \end{array}$ , where  $v$  is possibly identical to  $x$ .

**P11: Segment introduction in convex broken line:** From the diagram  $\delta[x_1, x_2, \dots, x_n]$ , deduce  $\gamma[x_1, x_2, \dots, x_n]$ , where  $\delta[x_1, x_2, \dots, x_n]$  is a convex broken line, and  $\gamma[x_1, x_2, \dots, x_n]$  is the diagram which results from joining two points  $x_i$  and  $x_j$  in  $\delta[x_1, x_2, \dots, x_n]$ . A broken line is defined to be a sequence of rays and/or segments such that no two are collinear, and such that each pair that are adjacent in the sequence share a common endpoint. Such a broken line is considered to be convex if the area it bounds is convex; that is, if no piece of the line crosses the extension of any other piece of the line.

**P12: Parallel Postulate:**

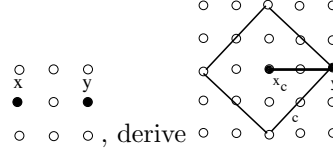


From  $\begin{array}{c} \circ \quad \circ \quad \circ \quad \circ \quad \circ \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}$  and  $\begin{array}{c} \circ \quad \circ \quad \circ \quad \circ \quad \circ \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}$  along with the metric assertion

$zxv < \text{angle } wyx$ , derive  $\begin{array}{c} \circ \quad \circ \quad \circ \quad \circ \quad \circ \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \\ \circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}$ .

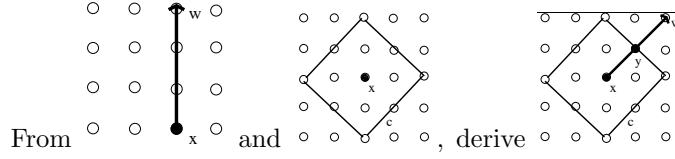
**P13: Extension of rays or segments:** Suppose  $x_1$  and  $x_2$  label a segment or ray  $l$  in  $\delta[\vec{x}]$  such that all linear elements of  $\delta[\vec{x}]$  either intersect  $l$ , or form part of a convex broken line which contains or intersects  $l$ .

Then from  $\delta[\vec{x}]$ , deduce  $\gamma[\vec{x}, y]$ , where  $\gamma[\vec{x}, y]$  is the result of extending  $l$  in  $\delta[\vec{x}]$  to a segment, ray, or line.



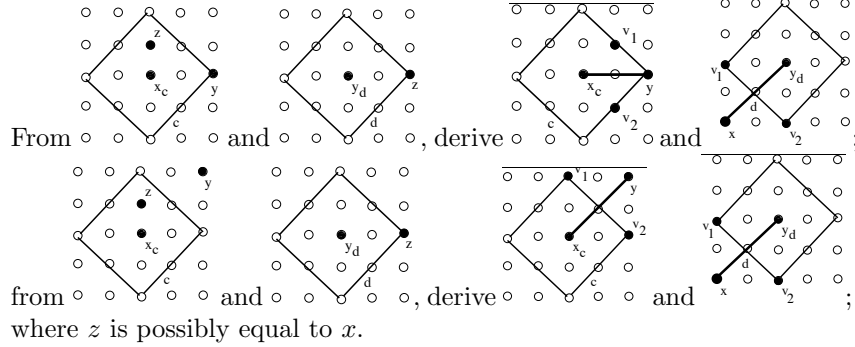
**P14: Circle Introduction:** From  $\circ \circ \circ$ , derive  $\circ \circ \circ$ .

**P15: Intersections of a ray and a circle:**



From  $\circ \circ \circ$  and  $\circ \circ \circ$ , derive  $\circ \circ \circ$ , where  $x$  is possibly the center of the circle.

**P16–P17: Intersections of a circle and a circle:**



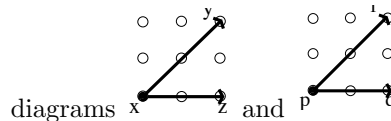
from  $\circ \circ \circ$  and  $\circ \circ \circ$ , derive  $\circ \circ \circ$  and  $\circ \circ \circ$ ; where  $z$  is possibly equal to  $x$ .

**P18: Modus Ponens:** From  $\Delta, A$  and  $\Delta, A \rightarrow \Gamma, B$ , deduce  $\Gamma, B$ .

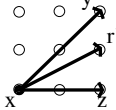
**P19: Diagram Combination:** From  $\Delta$  and  $\Sigma$ , deduce  $\Gamma$ , where  $\Gamma$  is a combination of  $\Delta$  and  $\Sigma$ , subject to the following conditions:

- There is a sub-diagram  $\Sigma'$  of both  $\Delta$  and  $\Sigma$  such that there is at most one object  $l$  in  $\Sigma$  not in  $\Sigma'$ ;
- $l$  is a point, ray, or line;
- $\Sigma$  contains at least one linear element;
- if  $l$  is a linear element, a point  $x$  in  $\Sigma'$  lies on  $l$  in  $\Sigma$ ,  $\Delta$  contains no circles, and all linear elements in  $\Delta$  either intersect  $x$  or else form part of a single convex broken line which intersects  $x$ ; and
- all possible combinations of  $\Delta$  and  $\Sigma$  other than  $\Gamma$  can be shown within the system to lead to a contradiction—that is, to a derivation of  $\perp$  or of a diagram inconsistent with one already derived.

**P20: Angle Trichotomy:** Suppose that we have already derived the

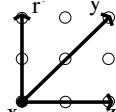


diagrams  $x \rightarrow z$  and  $p \rightarrow q$ . We can then break our proof into three cases with additional premises:



**Case 1:** we assume the diagram  $x$   $z$   $r'$  and the metric assertion  $qpr =_{\text{angle}} zxr'$  along with all previously derived diagrams and metric assertions;

**Case 2:** we assume the metric assertion  $qpr =_{\text{angle}} zxy$  along with all previously derived diagrams and metric assertions; and



**Case 3:** we assume the diagram  $x$   $z$  and the metric assertion  $qpr =_{\text{angle}} zxr'$  along with all previously derived diagrams and metric assertions.

If we can then derive a contradiction in two of these three cases, we can conclude the metric assertion added in the third case, and add it to the main branch of our proof.

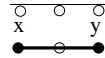
### A.3 Quantitative Rules

**Q1: Metric inferences from the context:** From diagrams  $\Gamma_1 \dots \Gamma_k$ , derive metric assertion  $A$ , where  $A$  was derived during the construction stage, and every object name that occurs in  $A$  labels a geometric object in one of the diagrams  $\Gamma_1 \dots \Gamma_k$ .

**Q2: Metric Assertion Conjunction:** From metric assertions  $A$  and  $A'$ , derive  $A \& A'$ .

**Q3: Atomic Metric Assertion Extraction:** From  $A$ , derive  $A'$ , where  $A$  is a conjunction of atomic metric assertions, and  $A'$  is one of the conjuncts.

**Q4: Equality inferences—Reflexivity:**



From  $\circ \circ \circ$ , derive the metric assertion  $xy =_{\text{seg}} xy$ ; from  $x$   $y$   $z$ , derive  $xyz =_{\text{angle}} xyz$ ; from a diagram containing a polygon labeled  $x_1 \dots x_n$ , derive the assertion  $x_1 \dots x_n =_{\text{area}} x_1 \dots x_n$ . (Note that the rule for angles given here doesn't agree with the given diagram if we follow the usual convention that angles are named with their center vertex in the middle; this is probably a typo in the original.)

**Q5: Equality inferences—Symmetry:** From a metric assertion of the form  $s = t$ , derive  $t = s$ .

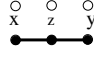
**Q6: Substitution:** From a metric assertion of the form  $s = t$  and another metric assertion  $A$ , derive  $A[s/t]$  ( $A$  with the variable  $s$  substituted for  $t$ ).

**Q7: Orientation independence of magnitude:** From  $xy =_{\text{seg}} zw$ , derive  $yx =_{\text{seg}} zw$ ; from  $xyz =_{\text{angle}} zwv$ , derive  $xzy =_{\text{angle}} zvw$ ; and from  $x_1 \dots x_n =_{\text{area}} y_1 \dots y_k$ , derive  $x_n x_1 \dots x_{n-1} =_{\text{area}} y_1 \dots y_k$ . (Again, the rule for angles doesn't make sense as written if we follow the usual convention that angles are named with their center vertex in



the middle. What is probably meant is: from  $xyz =_{\text{angle}} twv$ , derive  $zyx =_{\text{angle}} twv$ .)

**Q8: Inequality inferences—< introduction:**



From  $\circ \circ \circ$ , derive the metric assertion  $xz <_{\text{seg}} xy$ ; from  $\triangle xzy$ , derive the metric assertion  $xzy <_{\text{angle}} xzw$ ; and from  $\delta[poly(x_1, \dots, x_n)]$ , derive the metric assertion

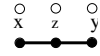
$$x_1 \dots x_k <_{\text{area}} x_1 x_{k+1} x_{k+2} \dots x_n,$$

where there are diagrams  $\sigma[poly(x_1, \dots, x_n)]$  and  $\gamma[poly(x_1 x_{k+1} x_{k+2} \dots x_n)]$  whose polygons are contained by the polygon of  $\delta[poly(x_1, \dots, x_n)]$ . (Note that the rule for angles again appears to be misstated; the conclusion should presumably be  $zxy <_{\text{angle}} xzw$ . Furthermore, the rule for area of polygons given here also appears to be incorrect; if two polygons are contained in a larger polygon, we have no way of knowing which of the two is larger. Probably what is intended is the conclusion  $x_1 \dots x_k <_{\text{area}} x_1 x_2 \dots x_n$ .)

**Q9: Inequality inferences—Transitivity:** From  $s < t$  and  $t < r$ , derive  $s < r$ .

**Q10: Inequality inferences—Strictness of <:** From  $s < s$ , derive  $\perp$ .

**Q11: Summation of magnitudes:**



From  $\circ \circ \circ$ , derive the metric assertion  $xz + zy =_{\text{seg}} xy$ ; from



$\triangle wxy$ , derive the metric assertion  $wxy + yxz =_{\text{angle}} wxz$ ; and from  $\delta[poly(x_1, \dots, x_n)]$ , derive the metric assertion

$$x_1 \dots x_k + x_1 x_{k+1} x_{k+2} \dots x_n =_{\text{area}} x_1 \dots x_n,$$

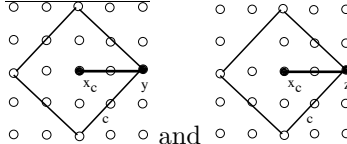
where there are diagrams  $\sigma[poly(x_1, \dots, x_n)]$  and  $\gamma[poly(x_1 x_{k+1} x_{k+2} \dots x_n)]$  whose polygons are contained by the polygon of  $\delta[poly(x_1, \dots, x_n)]$ .

**Q12: Commutativity:** From  $s + t = s + t$ , derive  $s + t = t + s$ .

**Q13: Equals subtracted from equals are equal:** From  $s + r = t + r$ , derive  $s = t$ .

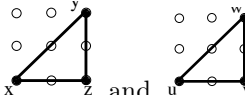
**Q14: Halves of equals are equal:** From  $s + s = t + t$ , derive  $s = t$ .

**Q15: Points on circle are equidistant from center:**



From  $\circ \circ \circ$  and  $\circ \circ \circ$ , derive  $xy =_{\text{seg}} xz$ .

**Q16: Side angle side congruence:**



From the diagrams  $\triangle xyz$  and  $\triangle uvw$  and the metric assertions

$xy =_{\text{seg}} uv$ ,  $xz =_{\text{seg}} uw$ , and  $xyz =_{\text{angle}} uvw$ , derive any of the following conclusions:  $yz =_{\text{seg}} vw$ ,  $zxy =_{\text{angle}} vuw$ ,  $yzx =_{\text{angle}} wvu$ , or  $xyz =_{\text{area}} uvw$ . (Note that once again the hypotheses here seem to be misstated relative to the given diagrams. The intended hypotheses are probably  $xy =_{\text{seg}} uv$ ,  $yz =_{\text{seg}} vw$ , and  $xyz =_{\text{angle}} uvw$ . However, this would mean that this rule will only apply to triangles whose given side, angle, and side lie in the same orientation as one another. This is an issue, because Euclid sometimes applies the Side Angle Side rule to triangles whose orientations are reversed from one another. )

**Q16: Empty assertion introduction:** The empty metric assertion  $\Theta$  can be introduced at any point in a derivation without any premises.

### Appendix B Provable claims in **Eu**

**Claims** play the role in **Eu** that theorems play in most sentential proof systems; they are the type of objects that **Eu** proves. **Eu** recognizes two kinds of claims: atomic claims, and conditional claims. An atomic claim consists of a diagram  $\Delta$  along with a metric assertion  $A$ ; we indicate that such an atomic claim is provable in **Eu** using a turnstile symbol, as follows:  $\vdash_{\mathbf{Eu}} \Delta, A$ . If  $\Delta, A$  and  $\Gamma, B$  are atomic claims, then  $\Delta, A \rightarrow \Gamma, B$  is a conditional claim. Again, we indicate that a conditional claim is provable in **Eu** using the turnstile symbol:  $\vdash_{\mathbf{Eu}} \Delta, A \rightarrow \Gamma, B$ . Claims can be proven in **Eu** using the following rules:

**Diagrams without circles:** If  $\Delta$  is a diagram in which no circles appear, then  $\vdash_{\mathbf{Eu}} \Delta, \Theta$ , where  $\Theta$  is the empty metric assertion.

**Substitution:** If  $\vdash_{\mathbf{Eu}} \Delta, A$ , then  $\vdash_{\mathbf{Eu}} \Delta[x/y], A[x/y]$ ; likewise, if  $\vdash_{\mathbf{Eu}} \Delta, A \rightarrow \Gamma, B$ , then  $\vdash_{\mathbf{Eu}} \Delta[x/y], A[x/y] \rightarrow \Gamma[x/y], B[x/y]$ . (Mumma does not put any restrictions on the substitutions that can be made, but probably we should require that  $x$  can only be substituted for  $y$  in a claim if  $x$  doesn't already occur as a variable in any metric assertion of the claim or as a label in any diagram of the claim.)

**Modus Ponens:** If  $\vdash_{\mathbf{Eu}} \Delta, A$  and  $\vdash_{\mathbf{Eu}} \Delta, A \rightarrow \Gamma, B$ , then  $\vdash_{\mathbf{Eu}} \Gamma, B$ .

**Derivation:** If, starting with the diagram  $\Delta$  and metric assertion  $A$  as assumptions, we can derive the diagram  $\Gamma$  and the metric assertion  $B$  in the demonstration stage of a proof in **Eu**, using the rules of **Eu** summarized in Appendix A, then  $\vdash_{\mathbf{Eu}} \Delta, A \rightarrow \Gamma, B$ .

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