A Brief Proof of the Full Completeness of Shin's Venn Diagram Proof System

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Abstract

In a article in the Journal of Philosophical Logic in 1996, "Towards a Model Theory of Venn Diagrams," (Vol. 25, No. 5, pp. 463–482), Hammer and Danner proved the full completeness of Shin's formal system for reasoning with Venn Diagrams. Their proof is eight pages long. This note gives a brief 5 line proof of this same result, using connections between diagrammatic and sentential representations.

In [4], Shin defines a semantics for Venn Diagrams and a formal system for manipulating them, and proves the following theorem:

Theorem 1 (Finite Completeness) *If Δ is a finite set of diagrams, D is a single diagram, and $\Delta \models D$, then $\Delta \vdash D$.*

In [2], in this Journal (reprinted in [3]), Hammer and Danner extend this result to infinite sets of diagrams, proving:

Theorem 2 (Full Completeness) *If Δ is any (possibly infinite) set of diagrams, D is a single diagram, and $\Delta \models D$, then $\Delta \vdash D$.*

Their proof is entirely diagrammatic, and is essentially a Henkin-style satisfiability argument, necessarily made more complicated because these Venn diagrams have no way of expressing negation. To have given a diagrammatic proof of this theorem is an impressive achievement, and their proof is eight pages long. However, by exploiting the connections between diagrammatic and sentential representations, we can give a much shorter, five line proof of this theorem.

Arguments like this that connect diagrammatic and sentential representations are sometimes referred to as "heterogeneous reasoning." See, for example, [1]. The brevity of the current proof illustrates how fruitful such an approach can be.

Theorem 2 would follow directly from Theorem 1 if we had a diagrammatic analogue of the following form of the Compactness Theorem:

Theorem 3 (Compactness) *If T is a (possibly infinite) set of first-order sentences, ψ is a first-order sentence, and $T \models \psi$, then $U \models \psi$ for some finite subset U of T.*

Mimicking a proof of this theorem in the diagrammatic system is quite difficult, though, because most proofs rely heavily on negation, which isn't available in the diagrammatic syntax. (Hammer and Danner essentially find a way to fake negation well enough to make their proof go through.) However, we don't need to prove it directly in the diagrammatic setting, because we can translate the diagrams into first order sentences and then apply the Compactness theorem for first-order logic. The translation is given by the function f from the following theorem:

Theorem 4 (Diagrams to Sentences) *Let Γ be a set of diagrams and let L be a first-order language containing at least as many relation symbols as there are equivalence classes of basic regions in Γ. Then there exists a function f from diagrams in Γ to sentences of L such that for all models M and diagrams $D \in \Gamma$, $M \models D$ iff $M \models f(D)$.*

This function is discussed in chapter 5 of [4].

Theorem 2 follows directly from Theorems 1, 3, and 4, as follows:

Assume that Δ is some (possibly infinite) set of diagrams, D is a single diagram, and $\Delta \models D$. Let $\Gamma = \Delta \cup D$, and let $T = \{f(D_i) | D_i \in \Delta\}$. By Theorem 4, $T \models f(D)$, so by Theorem 3, there is a finite subset U of T such that $U \models f(D)$. Let $\Theta = \{D_i \in \Delta | f(D_i) \in U\}$. Using Theorem 4 again, we find that this means that $\Theta \models D$, so by Theorem 1, $\Theta \vdash D$, and so since Θ is a subset of Δ, this means that $\Delta \vdash D$, proving Theorem 2.

The brevity of this proof shows the power of exploiting the connections between diagrammatic and sentential representations.

References

[1] Barwise, Jon and John Etchemendy,"Heterogeneous Logic", in *Logical Reasoning with Diagrams*, Gerard Allwein and Jon Barwise, eds., New York: Oxford University Press, 1996.

[2] Hammer, Eric and Norman Danner,"Towards a Model Theory of Venn Diagrams", *Journal of Philosophical Logic*, Volume 25, No. 5, 1996, pp. 463–482.

[3] Hammer, Eric and Norman Danner,"Towards a Model Theory of Venn Diagrams", in *Logical Reasoning with Diagrams*, Gerard Allwein and Jon Barwise, eds., New York: Oxford University Press, 1996.

[4] Shin, Sun-Joo, *The Logical Status of Diagrams*, Cambridge: Cambridge University Press, 1994.