

¹ Edited: 7:06 A.M., November 4, 2011

JOURNAL OF INQUIRY-BASED LEARNING IN MATHEMATICS

Notes for R H Bing's Plane Topology Course

Nathaniel Miller << Sam Creswell << R H Bing

Nathaniel Miller, University of Northern Colorado
Sam Creswell, San Antonio College
R H Bing, University of Wisconsin

¹The relation $X \ll Y$ indicates that these notes were originally authored by Y and subsequently prepared for submission by X without significant modification and with appropriate permissions. While Y is credited with the genesis of these notes, X takes full responsibility for the current version. In this case, it indicates that these notes were originally authored by R H Bing. They were edited by Sam Creswell, who corrected some errors, added his own introduction, and modernized some of the terminology. The current version was prepared by Nathaniel Miller, who further edited the notes, changed the formatting, and added additional introductory material and teaching notes for the instructor.

Contents

Acknowledgements	v
History of Changes to these Notes	vi
Introduction to the Instructor by Nathaniel Miller	viii
Introduction by Sam Creswell	xii
Introduction to the Instructor by R H Bing	xiv
Introduction to the Student	xvi
1 Operations with sets	1
2 Limit Points	4
2.1 Theorems	6
2.2 Questions	7
3 Connectedness	8
3.1 Theorems	9
3.2 Exercises	9
4 The Real Line	10
4.1 Theorems	11
4.2 Questions	11
5 Sequential Compactness	12
5.1 Theorems	12
5.2 Questions	13

6	Countable Collections	14
6.1	Exercises	14
6.2	Theorems	14
7	Coverings	16
7.1	Exercises	16
7.2	Theorems	16
8	Converging Sequences	18
8.1	Exercises	18
8.2	Theorems	19
9	Continuous Functions	20
9.1	Theorems	21
9.2	Exercises	22
10	Uniform Continuity	23
10.1	Exercises	23
10.2	Theorems	23
11	Sequences of Functions	24
11.1	Exercises	24
11.2	Theorems	24
12	Slopes	26
12.1	Exercises	27
12.2	Theorems	27
13	Transformations	29
13.1	Exercises	30
13.2	Theorems	31
14	Function of a Function	33
14.1	Theorem	34
15	Homeomorphisms	35
15.1	Exercises	37

15.2 Theorems	37
16 Cantor Set	39
16.1 Exercises	39
16.2 Theorems	39
17 Baire property	41
17.1 Theorems	41
A Nathaniel Miller's Sample Midterm	44
B Nathaniel Miller's Sample Final	46

Acknowledgements

These notes were originally authored by R H Bing. They were edited by Sam Creswell, who corrected some errors, added his own introduction, and modernized some of the terminology. The current version was prepared by Nathaniel Miller, who further edited the notes, changed the formatting, and added additional introductory material and teaching notes for the instructor.

R H Bing's notes are made available through permission of the copyright holder :

The Educational Advancement Foundation
2303 Rio Grande
Austin, Texas 78705
(512)469-1700
FAX (512)469-1790
info@edu-adv-foundation.org

History of Changes to these Notes

The following changes were made by Nathaniel Miller in updating Sam Creswell's version of these notes:

- The notes were put into the *Journal of Inquiry-Based Learning in Mathematics* style.
- The section *Introduction to the Instructor by Nathaniel Miller* was added.
- Appendices A and B containing a sample midterm and final were added.
- Several *Notes to the Instructor* were added throughout the text as end-notes.
- Definitions were set out from the rest of the text and numbered, to make them easier to refer to. Likewise, Axiom 3 was given a number.
- Theorems, Exercises, and Questions were put into separate sections of each chapter.
- An additional circle was added to the illustration of Axiom 1 in Figure 2.2, to reflect the fact that W is contained in the intersection of U and V , but needn't be all of the intersection. (This also agrees with the illustration in the previous version of the notes.)
- Definition 3.6, which was added by Sam Creswell in his version of these notes, was changed to agree with Bing's definition of component given in [2]. This also makes Exercise 3 in Section 3.2 make more sense. It was moved earlier in the text in order to be consistent with the other chapters, in which all definitions are given before any theorems or questions.
- The definitions in Chapter 12 were slightly modified.
- Some other formatting and notation was changed, and other small changes were made.

The following is a partial list of some of the changes made by Sam Creswell in the previous version of these notes:

- The section *Introduction by Sam Creswell* was added.
- The property that Bing referred to as “compactness” was renamed “sequential compactness,” while the property that Bing referred to as “bi-compactness” was renamed “compactness,” in order to bring the notes into agreement with modern usage.
- A definition for the term “component” was added, since this was missing from the previous version of the notes.
- New figures were produced to replace the old hand-drawn figures.

Introduction to the Instructor by Nathaniel Miller

These course notes are R H Bing's course notes for an introductory course in elementary point set topology, designed to be taught through an inquiry-based method in which the students in the course prove and present the central course material. This method of teaching, often referred to as the "Moore Method," was pioneered at the University of Texas by R. L. Moore. Since Bing was a student of Moore's and he states in his introduction that much of the material in these notes was also in Moore's class at the University of Texas, we have to assume that these notes are close to the notes that Moore himself would have used.

Bing also wrote a long expository article on elementary point set topology which appeared in 1960 as a special issue of the *American Mathematical Monthly* [2]. In this article, he discusses and tries to motivate many of the same questions found in these course notes. I usually give this article to my students at the end of the semester. I think it would be of interest to anyone planning to teach out of these notes.

In preparing these notes for submission to the *Journal of Inquiry-Based Learning in Mathematics*, I have only made minor changes to the course sequence passed down from R H Bing and edited by Sam Creswell. A detailed list of my changes appears in the previous section. I have tried to update the formatting, and have added some additional notes on teaching from these materials, drawn from my own experiences using them. Some of these appear as numbered endnotes in this Instructor's version of these materials, in the section *Notes to the Instructor*, which appears following the main text.

I have taught classes from these materials twice. I have to say, they work wonderfully. They reflect a course that is already well refined, and they have been used successfully by many people over the years.

The class that I have taught from these notes is a Master's level topology course at the University of Northern Colorado. This class is primarily taken

by Master's students planning to continue into our Ph.D. program in mathematics education, along with some advanced undergraduates. The first time I taught this course, it had nine students; the second time, it had five. I think that 10 students is probably close to the ideal size for this class. With my classes, we were able to cover roughly the first five chapters of these notes in one semester. As Bing points out in his Introduction to the Instructor, the notes contain much more material than most classes will be able to cover in one semester. In fact, my classes took almost half the semester on Chapter 2 alone.

There are many possible ways to structure the mechanics of how a class using these notes is run. One great source of information for planning a Moore Method or Modified Moore Method course is the book by Coppin, Mahavier, May, and Parker [3], in which several experienced practitioners of this method discuss how they structure their own classes. I would encourage anyone considering teaching a Moore Method or Modified Moore Method course for the first time out of these notes to take a look at this book.

My classes were taught using a Modified Moore Method in which students worked on the problem sequence outside of class and presented their solutions at the board in class, as would happen in the most traditional Moore Method. However, I also collected a weekly homework assignment, in which students turned in one problem of their choice each week, and I had an in-class midterm and a take-home final. See Appendix A for a sample midterm, and Appendix B for a sample final exam.

I also allow students to collaborate outside of class. I do not permit students who have already solved a problem to share their solutions with other students, but students who have not solved the problem are permitted to work together in solving it. This policy has worked well for me, and I have not had any problems with students violating my rule. I also haven't had many instances in which weak students worked with strong students in order to get answers from them, because strong students don't generally choose to work with someone if they don't expect them to be helpful in solving the problem.

The following are some of my course policies, taken from the syllabus I hand out on the first day.

An Introduction to Math 540: Topology

Welcome to math 540! This course will probably be very different from most other math courses that you have taken. We're going to be following

a method of learning mathematics pioneered by the mathematician R. L. Moore at the University of Texas at Austin, and variously known as “The Moore Method” or “The Texas Method.” The hallmark of this method is that the students (you!) are responsible for creating most of the mathematics discussed in the course.

The core of the course will be a sequence of definitions, problems to be solved, and theorems to be proven. You can think of it as a kind of a game.

The Rules of the Game:

1. The object of the game is to collect points.
2. The main way of getting points is by solving Problems and proving Theorems.
3. Problems and Theorems should be viewed as *conjectures*—that is, they may be either true or false. They can be solved by showing which category they fall in. In other words, you solve a problem or theorem by giving a convincing argument that it is true (or false). If the Problem or Theorem is not correct as stated, you should try to see if there is a way to fix it.
4. There are two ways of earning points by solving problems:
 - (a) The best way to earn points is to present a correct solution to an unsolved problem to the class. A solution is correct if it is unanimously judged by the class to be completely convincing.
 - (b) I will also ask you to turn in a written solution to one problem once a week. Solutions that are correct the first time they are submitted are worth 10 points.
 - (c) Written solutions that are incomplete in some way may be revised; points will only be awarded for complete, correct solutions. Solutions that need to be revised will be worth one less point for each time they have to be revised.
5. Points may be awarded for other reasons, such as for proposing a conjecture that gets made into one of the course problems.
6. The intent of this course is for you to solve the problems yourself. You may discuss the problems you are working on with other members of the class as long as neither of you has already solved the problem. If you have solved a problem you can try to help another student arrive at the solution themselves, but you MAY NOT just give them your solution.

7. There is no course textbook, and the *only* texts that you may consult are the official course notes and your own notes. You should not consult any textbooks unless I have specifically given you permission. Solutions to some of the problems that we will be covering can be found in textbooks, but if you consult these books, you will be depriving yourself of the understanding that will come from solving the problems yourself.
8. We will also have an in class midterm and a take home final at the end of the course. We will meet at our assigned final meeting time to turn in and present the take home final.
9. A class participation score will also be added to your final point total; this score will be subjectively determined by me, and will reflect how actively and productively you participated in all aspects of the class.
10. Grades will be determined by how many points you have at the end of the course. I can't tell you in advance how many points will correspond to a certain grade, but if I feel that anyone with a certain number of points has been doing A level work, for example, then everyone with at least that many points will get an A.
11. These rules may change as the semester progresses.
12. Don't forget to have fun!

The notes that we will be using were written by R H Bing. R H Bing was a high school math teacher who went back to school at the University of Texas at Austin to get a Master's Degree. While there, he met and took classes from R. L. Moore. These classes made him want to go on and become a research mathematician. He got his Ph.D. in topology as a student of R. L. Moore's, and went on to a distinguished career in mathematics. He taught for many years at the University of Wisconsin, and served at different times as President of both the American Mathematical Society and the Mathematical Association of America.

Introduction by Sam Creswell

R H² Bing (1914 – 1986), a native of Oakwood³, Texas, was a 20th century mathematician and teacher of mathematics.

Bing received a bachelor's degree from Southwest Texas State Teachers College (now, Texas State University - San Marcos) and began teaching and coaching football in Texas public schools. He took a summer course under R. L. Moore, who recognized his immense talents, encouraged him to work toward a doctorate, and provided a teaching position for him while he was in graduate school. Upon receipt of his doctorate from Moore, Bing accepted a position at Wisconsin and soon cemented his reputation as a teacher and researcher. Late in his career, Bing returned to Texas, where he served as the chair of the department, a position which he once held at Wisconsin.

Bing served as a President of both the Mathematical Association of America and the American Mathematical Society. Further, he was member of the National Academy of Sciences.

In common with his mentor, R. L. Moore, Bing was passionately interested in undergraduate mathematics. As Moore found immense talent in a Texas high school football coach, Bing was also interested in recognizing and developing talent. Few mathematicians with Bing's prominence attempt to discover talent among those whose mathematical background is modest. These notes manifest his interest in doing so.

A partial list of his accomplishments may be found on page 220 of [1]. Further, [1] provides much more information about Bing, his mentor, and inquiry-based learning in mathematics, as implemented at Texas.

As these notes passed through the hands of many instructors and typists over the years, a handful of typographical errors naturally developed. In consultation with the editors of this journal, we have made corrections, as necessary.

²Bing had no given names, as such. His first name was the letter 'R' and his second name was the letter 'H' (see p. 208 [1]). Bing once directed some painters at Wisconsin to produce some signage for him writing down 'R only H only Bing'. They duly produced signage saying 'Ronly Honly Bing'!

³Visits to Oakwood support the observation that mathematical talent may be found anywhere. The Bings who live in Oakwood today are aware of their kinsman's prowess and are proud of his achievements.

Sam Creswell
Mathematics Department
San Antonio College
San Antonio, Texas 78212-4299
screswell@alamo.edu

Introduction to the Instructor by R H Bing

These are notes for a one-semester undergraduate course started by R H Bing at the University of Wisconsin. R. E. Fullerton, S. C. Kleene, and R. F. Williams subsequently taught the course at Wisconsin. These notes were used by C. B. Allendoefer at the University of Washington and by W. L. Duren, Jr. at Tulane University. Much of the material in these notes is in a year course (Math 24) at the University of Texas offered by R. L. Moore.

The course emphasizes the part of plane topology used in calculus. In general, the course is easier for those who have had a course in advanced calculus. Conversely, those who took the plane topology course and subsequently took advanced calculus reported that the topology course was good preparation for advanced calculus. The course is primarily designed for seniors. It would seem to be good either as a terminal course or as a preparation for graduate work. The course has been offered to high school teachers in summer school. The course should add considerably to the students' vocabulary since some of the more prevalent terms in topology are used.

The amount of material covered in the course varies depending on the ability of the students and the number of diversionary topics pursued. The notes contain about enough material for a year's work so most classes will cover only about one half of the theorems. It is felt that it is better for the students to learn some of the material thoroughly and have confidence in what they know than to cover it all. Perhaps some of the more energetic students will pursue some of the interesting theorems not covered in class after the course is over. Many of the topics are covered in other courses but the emphasis there is often algebraic rather than geometric to topological.

The theorems in the course were proved by the students. Those that they could not prove were postponed. Students who had little previous experience in proving theorems were slow in getting started so some easy theorems were included at the first to help them begin. The students presented the theorems at the blackboard (preferably without notes) under the critical eye of their fellow students and with the friendly criticism (but without help) from the instructor. As the students gain experience in proving and

presenting theorems and as they get a taste of the thrill of accomplishment, the pace should quicken.

The instructor's lecturing was confined to discussing the definitions in the notes, relevance of these definitions and corresponding theorems, various unsolved problems in topology, and other topics not in the notes. The theorems that one obtains by one's own efforts by pondering, trial and error, and other methods are burned deeper into one's consciousness than those presented by others. Consequently, if a student was still working on a problem when it was presented in class and wanted to complete it alone, he was permitted to absent himself from class during the presentation and report later outside of class on his progress. Some of the slower students did not prove many theorems but they participated in class by recalling definitions and taking part in "brain storming expeditions" where theorems were conjectured. They were expected to learn the proofs given by the better students. All students were advised to keep notebooks where they wrote up the proofs in their individual styles.

This course reversed the usual tradition of the instructor presenting the fundamental material and the students doing drill to help assimilate it. Here the student did the fundamental work and the instructor supplemented it.

Introduction to the Student

In this course we study point sets in the plane, properties of these sets, and operations with these sets. Special emphasis will be placed on the aspects of plane topology that are used in calculus.

Rather than trying to consider many theorems superficially, only a selected group will be studied. It is felt that if a person understands these, he can prove many others.

Before proofs are given in class, the student will be given all opportunity to work them out himself. The more proofs that the student can develop himself, the more mathematical training he obtains. In fact, much original research has developed because a mathematician, in developing for himself the proof of a known theorem, finds new methods that can be used elsewhere. In proving a theorem a person may learn much more than just the steps in the proof. He may learn which methods and tricks will work and which will not (this is important too) in similar situations.

Mathematics may be thought of as a way of thinking rather than a collection of facts. The person who can develop a proof for a theorem or work out a solution of a problem is regarded more highly as a mathematician than the person who only remembers many mathematical facts and names of various theorems.

In this course, emphasis will be placed on mathematical thinking. Although there will be many mathematical truths developed, importance will be given to the development rather than to the facts themselves. This is not to say that facts of mathematics are not important. They are tools that enable us to do a better job of mathematical thinking and should be learned well.

Chapter 1

Operations with sets

We need to acquaint ourselves with the following terminology, undefined terms, and definitions of topology.

Point. In synthetic geometry and in topology, a point is usually regarded as an undefined term.

In this course, we suppose that the points under consideration lie in the Euclidean plane (unless specified otherwise). We will feel free to use the properties of the plane learned in plane geometry and analytics.

In courses where the number system is developed first, a point is sometimes defined in terms of numbers. For example, in analysis, a point may be defined to be a complex number; in plane analytics, a point is an ordered pair of real numbers; in 3-dimensional Euclidean analytic geometry, a point is an ordered triple of numbers; and in 4-dimensional Euclidean analytic geometry, a point is an ordered quadruple of numbers.

Definition 1.1. A *point set* is a collection such that each element of the collection is a point.

If a collection contains only one element, there is a distinction between the collection and the element, though for brevity, we do not always emphasize this.

Definition 1.2. The *null set* is a collection with no element.

There is some objection to introducing such an artificial set. We have no need for null sets in this course and in places where non-null is mentioned we could state the results more simply if the convention were followed that each set contains a point. However, this convention is not generally followed.

Definition 1.3. $p \in X$ means that p is an element of the collection X . If p is not an element of X , we write $p \notin X$.

Definition 1.4. $X \subset Y$ means that each element of X is an element of Y . If $X \subset Y$, we say that X is a *subset* of Y . If X is a subset of Y but X is not all of Y , we say that X is a *proper subset* of Y . See Figure 1.1.

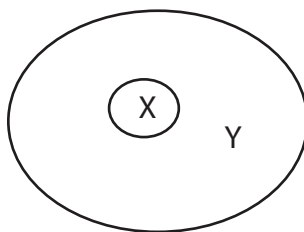


Figure 1.1: X is a Subset of Y .

Definition 1.5. $X \cdot Y$ or $X \cap Y$ (the *intersection*, *product*, or *common part* of X and Y) denotes the set of all points in both X and Y ; that is, $X \cdot Y = \{p \mid p \in X \text{ and } p \in Y\}$. See Figure 1.2.

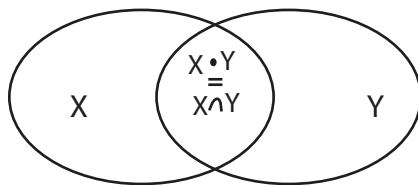


Figure 1.2: The intersection of X and Y .

Definition 1.6. $X + Y$ or $X \cup Y$ (the *sum* or *union* of X and Y) denotes the set of all points in either X or Y . That is, $X + Y = \{p \mid p \in X \text{ or } p \in Y\}$. See Figure 1.3.

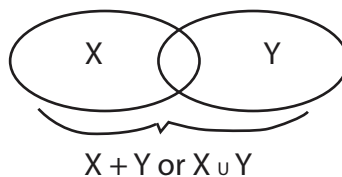


Figure 1.3: The sum of X and Y .

Definition 1.7. $X - Y$ (the *difference* of X and Y) denotes the collection of points of X that do not belong to Y . That is, $X - Y = \{p \mid p \in X \text{ and } p \notin Y\}$. See Figure 1.4.

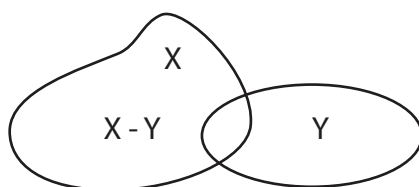


Figure 1.4: X Minus Y .

Chapter 2

Limit Points

Definition 2.1. A *neighborhood* is the interior of a circle.

Definition 2.2. A *neighborhood of a point* is a neighborhood containing the point. We do not suppose that the point is the center of the neighborhood.

As an extension of the notion of a neighborhood in other spaces with a distance function, a neighborhood is regarded as the interior of a generalized sphere.

Definition 2.3. The ε -*neighborhood of p* (written $N(p, \varepsilon)$) is the set of all points whose distance from p is less than ε .

We denote the distance between p and q by $\rho(p, q)$. Then $N(p, \varepsilon) = \{q \mid \rho(p, q) < \varepsilon\}$. If a point r lies in $N(p, \varepsilon)$, then $N(p, \varepsilon)$ is a neighborhood of r , although it may not be the ε -neighborhood of r . See Figure 2.1.

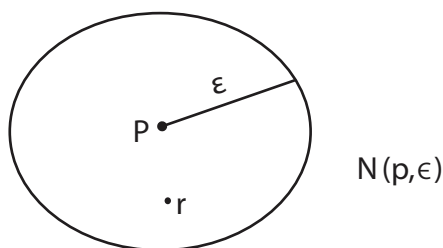


Figure 2.1: ε -Neighborhood of p .

If a space does not have a distance function, certain point sets in it may be designated as neighborhoods. A neighborhood of p is one of those designated point sets that contains p . These neighborhoods must satisfy certain conditions called axioms. Two axioms sometimes used are the following:

Axiom 1. If U and V are neighborhoods of the same point p , there is a neighborhood W of p such that $W \subset U \cdot V$.

Axiom 2 (Hausdorff or T_2 separation axiom). *If p and q are different points, there are neighborhoods N_p and N_q of p and q respectively such that N_p does not intersect N_q .*

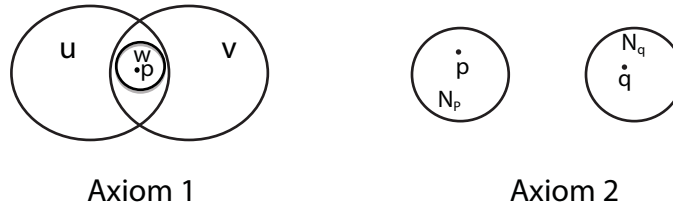


Figure 2.2: Axioms 1 and 2.

See Figure 2.2 for illustrations of these axioms. Sometimes weaker axioms are used but more frequently, stronger ones are employed.

Definition 2.4. If N is a neighborhood of a point p , we call $N - \{p\}$ a **deleted neighborhood of p** . Here $\{p\}$ denotes the collection whose only element is p .

Definition 2.5 (Open set). A point set D is **open** if each point p of D has a neighborhood that lies in D .

An open set can be expressed as a sum of neighborhoods, although the neighborhoods may be infinite in number. The interior of a square is open, but a disk (a circle plus its interior) is not.

Definition 2.6 (Neighborhood definition of limit point). The point p is a **limit point** of the point set M if each deleted neighborhood of p contains a point of M .

Definition 2.7 (ϵ -definition of a limit point). The point p is a **limit point** of the point set M if for each positive number ϵ there is a point q of M such that $0 < \rho(p, q) < \epsilon$.

Definition 2.8. The **boundary** of a set X is the set of all points p such that each neighborhood of p contains both a point of X and a point not of X .

Definition 2.9. The **closure** of a set X is the sum of X and all its limit points. This closure is designated by \overline{X} .

It follows from these definitions that the closure of X is equal to X plus its boundary.

Definition 2.10. A **closed set** is one that contains all its limit points.

Definition 2.11. The **complement** of a point set X is the collection of all points not in X .

2.1 Theorems

Prove the following theorems for the plane.

1. The sum of two open sets is open.
2. If p is a limit point of the sum of two closed sets, it is a limit point of one of them.¹
3. The sum of two closed point sets is closed.
4. The product of two closed sets is closed.
5. If a point lies in each of two neighborhoods, it lies in a neighborhood in the intersection of the two neighborhoods.
6. The product of two open sets is open.
7. If p is a limit point of A , each neighborhood of p contains infinitely many points of A .
8. For each set X , \overline{X} is closed.
9. For each set X , $\overline{X} = \overline{\overline{X}}$.
10. The boundary of a set is closed.
11. If p is a limit point of $A \cdot B$, it is both a limit point of A and a limit point of B .
12. The sum of a collection of open sets is open.
13. The intersection of a collection of closed sets is closed.
14. The closure of a set X is the intersection of all closed sets containing X .
15. The complement of a closed set is open.
16. The complement of an open set is closed.
17. The complement of $X + Y$ is the intersection of the complement of X and the complement of Y .
18. The complement of $X \cdot Y$ is the sum of the complement of X and the complement of Y .

2.2 Questions

Answer the following.

1. If X is a finite set of points on the x -axis, does X have a limit point?
2. If X is a subset of the x -axis and p is a limit point of X , does p belong to x -axis?
3. Is the interior of a circle the sum of an infinite collection of closed sets?
4. Name a property that is not possessed by the sum of some two sets even if it is possessed by each of them.²
5. Note that Theorem 7 does not state that there are infinitely many points that belong to each neighborhood. Explain why there may not be even one such point.³
6. Why may the sum of a collection of closed sets fail to be closed?
7. Why may the intersection of a collection of open sets fail to be open?
8. Name a plane set that is both open and closed. Name one that is neither.
9. Consider the set $M = \{(x, y) \mid 0 < x^2 + y^2 < 1 \text{ or } x = y = 1\}$. Which of the points $(0, 0)$, $(0, 1)$, $(1, 1)$, $(2, 0)$, and $(1/2, 0)$ are points of M , and which of them are limit points of M ?
10. If each of three sets contains a point of $A + B$, does this mean that either each of them contains a point of A or each of them contains a point of B ?
11. Which of the theorems in the first set shows that if p is not a limit point of X and is not a limit point of Y , then it is not a limit point of $X + Y$?
12. Show that plane neighborhoods satisfy the two axioms mentioned for abstract neighborhoods.

Chapter 3

Connectedness

Intuitively, we think that a set is connected if it is “all in one piece.” We use the following definitions.

Definition 3.1. Two sets are *disjoint* or *mutually exclusive* if neither contains a point of the other.

Definition 3.2. Two sets are *mutually separated* if neither contains either a point or a limit point of the other.

Each of the two mutually separated sets is relatively open and relatively closed in their sum.

Definition 3.3. A set A is *relatively open* (*relatively closed*) in Y if A is the intersection of Y and an open (closed) set.

A circle and its interior are mutually exclusive but they are not mutually separated. However, the interior of a circle and the exterior are mutually separated.

Definition 3.4. A set is *not connected* if it can be expressed as the sum of two non-null mutually separated sets.

Definition 3.5. A set is *connected* if it is not the sum of any two non-null mutually separated sets. See Figure 3.1.

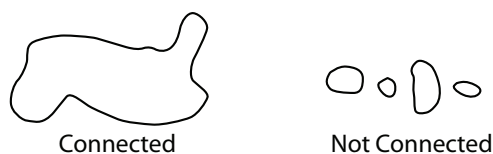


Figure 3.1: Connected vs. Not Connected

Definition 3.6. A *component* of a set X is a maximal connected subset of X . That is, it is a connected subset of X that is not properly contained in any larger connected subset of X .⁴

3.1 Theorems

Prove the following.⁵

19. If X is connected and Y is a subset of the boundary of X , then $X + Y$ is connected.
20. If a point p belongs to each element of a collection of connected point sets, their sum is connected.
21. If p is a point of the connected set M and $M - p$ is the sum of the mutually separated sets H and K , then $H + p$ is connected.
22. If p is a point of a point set X , then some component of X contains p .

3.2 Exercises

1. Show that the intersection of two connected sets may fail to be connected.
2. Is any finite set connected?
3. Name a property such that there is no set which is maximal with respect to having the property.

Chapter 4

The Real Line

Let \mathbb{R} be the number system with its ordinary topology. Then topologically \mathbb{R} is like a line. Here are three statements about \mathbb{R} that are of utmost importance.

1. \mathbb{R} is connected.
2. Each bounded monotone increasing sequence of elements of \mathbb{R} has a least upper bound.
3. \mathbb{R} satisfies the Dedekind cut property.

None of these statements about \mathbb{R} would be satisfied if we used the convention that only rational numbers are real. Hence we need an axiom of some sort in order to prove the properties. The one that we adopt is the following.

Axiom 3. *A straight line is connected.*

Definition 4.1. A set X in \mathbb{R} is **bounded** if there is a positive number m such that each element of X lies in the m -neighborhood of 0.

Definition 4.2. A sequence of numbers A_1, A_2, \dots is **monotone increasing** if each term of the sequence is larger than the one before it. **Monotone decreasing**, **monotone nonincreasing**, and **monotone nondecreasing** are defined in a similar fashion.

Definition 4.3. An **upper bound** of a collection X is a number that is greater than or equal to each element of X . The least such upper bound is called the **least upper bound** of X . Similarly, a **greatest lower bound** may be defined.

Definition 4.4. If X has a largest value, this value is called the **maximum** of X . If there is a value of X that is the largest value in some neighborhood, it is called a **relative maximum**. Similarly, a minimum may be defined.

Definition 4.5. The *Dedekind cut property* says that if \mathbb{R} is expressed as the sum of two non-null, mutually exclusive sets A and B , such that each element of A is less than each element of B , then either A has a maximum or B has a minimum.

Definition 4.6. If p and q are two points of a straight line, the *interval* $[pq]$ is the sum of p , q , and all points between them. We use (pq) to denote the *open interval* or *segment*. $(pq) = [pq] - (\{p\} + \{q\})$. Both $[pq]$ and (pq) are called *sects*.

4.1 Theorems

Prove the following.

23. A segment is connected. (Use Axiom 3 to show this.)
24. An interval is connected.
25. Each bounded monotone increasing sequence of numbers has a least upper bound. (Use Axiom 3 to show this. Theorem 25 itself is taken as an axiom in some calculus books.)
26. \mathbb{R} satisfies the Dedekind cut property.

4.2 Questions

1. Is the set of all points on the x -axis with rational abscissas (x values) connected?
2. If X is a bounded sequence of numbers and Y is the set of all numbers that follow infinitely many elements of X , is Y open, closed, or neither?

Chapter 5

Sequential Compactness

Definition 5.1. A set M is *sequentially compact* if for each infinite subcollection of M there is a point p of M which is a limit point of this subcollection.⁶

Definition 5.2. A set is *bounded* if it lies on the interior of some circle.

Definition 5.3. A sequence of sets A_1, A_2, \dots is called a *nested sequence* if for each positive integer i , $A_{i+1} \subset A_i$. A point is called a *common point* if it belongs to each A_i .

Suppose $f(x)$ is defined on an interval $(a \leq x \leq b)$. The notion of sequential compactness may be used in proving the following useful theorems from calculus which hold if f is continuous. See Chapters 9 and 10.

1. f is bounded.
2. f takes on a maximum and a minimum.
3. f is uniformly continuous.
4. $\int_a^b f(x)dx$ exists.

5.1 Theorems

Prove the following.

27. A sequentially compact set is closed.
28. An interval is sequentially compact. (Use Axiom 3 to show this.)
29. A square plus its interior is sequentially compact.

30. A bounded closed set is sequentially compact.
31. A sequentially compact set is bounded.
32. A closed bounded set has a highest point.
33. If p is a point and M is a closed set (non-null), there is a closest point of M to p .
34. (Bull's eye theorem). Each nested sequence of non-null sequentially compact sets has a common point.

5.2 Questions

1. Is the plane sequentially compact? The surface of a sphere?
2. Does each nested sequence of segments have a common point?

Chapter 6

Countable Collections

Definition 6.1. A collection is called *countable* if it has only a finite number of elements or if there is a one-to-one correspondence between its elements and the positive integers. A collection that is not countable is called *uncountable*.

6.1 Exercises

1. Give a plan for putting the rational numbers in a one-to-one correspondence with the positive integers.
2. Set up a one-to-one correspondence between the positive integers and the odd positive integers.
3. Set up a one-to-one correspondence between the points of a long interval and the points of a short interval.
4. Show that if M is a closed proper subset of an interval I that contains both ends of I , then $I - M$ is the sum of a countable collection of segments.
5. Does each uncountable subset of the x -axis contain a segment?

6.2 Theorems

Prove the following.

35. The rational numbers are countable.
36. The numbers between 0 and 1 are uncountable.

- 37. Each infinite collection has uncountably many subsets.
- 38. The sum of a countable collection of countable sets is countable.
- 39. No countable set with more than one point is connected.

Chapter 7

Coverings

Definition 7.1. A collection G of point sets is said to **cover** a set A if each point of A is contained in an element of G .

Definition 7.2. An **open cover** is a collection of open sets which covers a point set.

Definition 7.3 (compactness). A set H is **compact** if for each open covering G of M there is an open covering H with only a finite number of elements such that each element of H is a subset of an element of G . H is called a **finite refinement** of G .

7.1 Exercises

1. If G is a collection of segments of x -axis covering a segment, does a finite subcollection of G cover the segment?
2. If G is a collection of segments of x -axis covering an interval, does a finite subcollection of G cover the interval?
3. If G is a collection of intervals of x -axis covering an interval, does a finite subcollection of G cover the interval?
4. Is the interior of a circle compact?
5. Show that each finite set is compact.

7.2 Theorems

Prove the following.

40. An interval is compact. (Use the axiom.)
41. Each compact set is sequentially compact.
42. (Heine-Borel theorem). Each sequentially compact set is compact.
43. No interval is the sum of a countable number of closed sets no one of which contains an interval.

In certain abstract spaces, each compact set is sequentially compact but not conversely. In such spaces, Theorem 41 holds but 42 does not. In general, compactness is a stronger condition than sequential compactness and a person who uses it in the hypothesis of a theorem is not proving as strong a result as one who uses mere sequential compactness instead. However, this is not true in the plane. Some important generalizations of Theorem 43 are treated in Chapter 17 on the Baire property.

Chapter 8

Converging Sequences

Definition 8.1. A sequence of points $\{p_1, p_2, \dots\}$ (in which the p_i 's do not have to be distinct) is said to **converge** to p (written $\lim p_n = p$) provided that for each neighborhood D of p there is an integer n such that $p_n + p_{n+1} + \dots$ lies in D .

Definition 8.2. A sequence of numbers $\{A_1, A_2, \dots\}$ is said to **converge** to A provided that for each positive number ε there is an integer n such that no term of $\{A_n, A_{n+1}, \dots\}$ differs from A by more than ε .

Definition 8.3 (Cauchy Sequence). A sequence of points $\{p_1, p_2, \dots\}$ is called a **Cauchy sequence** if for each positive number ε there is an integer n such that no two of the points $\{p_n, p_{n+1}, \dots\}$ are at a distance from each other of more than ε . A Cauchy sequence of numbers is defined in a similar fashion.

Definition 8.4 (Limits). The **upper limit** for a set of numbers is the greatest possible limit of a sequence of numbers in the set. The **lower limit** of a set of numbers is defined similarly.

Definition 8.5 (Distance between sets). The distance between two sets is the greatest lower bound of the distances between the points of one set and the points of the other.

8.1 Exercises

1. Name a set of numbers with no upper limit.
2. Give two mutually exclusive closed point sets whose distance apart is 0.

8.2 Theorems

Prove the following.

44. Every Cauchy sequence of points on the x -axis converges.
45. Each Cauchy sequence of points in the plane converges.
46. Each bounded monotone increasing sequence of numbers converges.
47. Each bounded infinite sequence of points has a converging subsequence.
48. Each bounded infinite collection of numbers has an upper limit and a lower limit.
49. If X, Y are two bounded closed sets, there are points x, y of X, Y respectively such that $\rho(X, Y) = \rho(x, y)$.
50. If X, Y are mutually exclusive closed and bounded sets, they lie at a positive distance from each other.

Chapter 9

Continuous Functions

Definition 9.1. y is a *function* of x on the domain D , written

$$y = f(x) \ (x \in D),$$

provided that for each element of D there corresponds a number $f(x)$. We do not suppose that D is connected or that $f(x)$ is continuous. However, we do suppose that $f(x)$ is single-valued.

Definition 9.2. A point set G is a *graph* of some function $y = f(x)$ ($x \in D$) if it is the collection of all points p such that for some x of D , the coordinates of p are $(x, f(x))$. No vertical line intersects G in two or more points.

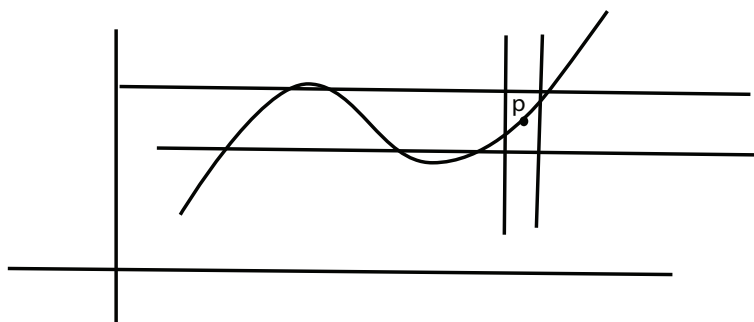
Definition 9.3 (Domain of definition). The *domain* of a function is the set of all values x for which the function is defined. It may be considered as the projection of the graph of the function on the x -axis. If the domain of definition of $f(x)$ is D , we may write $f(x)$ ($x \in D$).

Definition 9.4 (Range of dependent variable). The *range* of $f(x)$ ($x \in D$) is the set of all values y such that for some value x of D , $f(x) = y$. It may be considered as the projection of the graph of the function on the y -axis.

Definition 9.5. A function is said to be *bounded* if the range of the dependent variable is bounded.

Definition 9.6 (Continuity of a graph). The graph G of a function is *continuous* at the point p of G if for each pair of horizontal lines, one above and the other below p , there is a pair of vertical lines, one to the left and one to the right of p such that every point of G between the vertical lines is between the horizontal lines. See Figure 9.1.

We give two definitions of continuity of a function at a point.

Figure 9.1: The Continuity of the Graph G at p .

Definition 9.7 (Neighborhood definition of continuity at a point). The function $f(x)$ ($x \in D$) is continuous at the value x_0 of X if for each neighborhood N of $f(x_0)$, there is a neighborhood U of x_0 such that $f(y) \in N$ if $y \in U \cap D$.

Definition 9.8 (ε, δ definition of continuity at a point). The function $f(x)$ ($x \in D$) is continuous at the value x_0 of D if for each positive number ε there is a positive number δ (δ is a function of x_0 and f) such that if y belongs to D and differs from x_0 by less than δ , then $|f(y) - f(x_0)| < \varepsilon$.

Definition 9.9 (Continuity of a function). A function is continuous if it is continuous at each value of its domain of definition.

Definition 9.10 (Discontinuity). A function is *discontinuous* if it fails to be continuous at some value of its domain. (Some calculus books call a function discontinuous if its domain of definition is less than all the reals, but we do not follow this convention.)

9.1 Theorems

Prove the following.

51. A continuous function on a sequentially compact domain is bounded.
52. A continuous function on a sequentially compact domain assumes its maximum.
53. The graph of a continuous function on a closed domain is closed.
54. If $f(x)$ and $g(x)$ are continuous functions on the same domain of definition D , then $f(x) + g(x)$ is continuous on D .
55. If $f(x)$ ($0 \leq x \leq 1$) is discontinuous at each irrational value, it is discontinuous at some rational value.

56. Show that if $f(x)$ ($a \leq x \leq b$) and $g(x)$ ($a \leq x \leq b$) are continuous, $f(a) < g(a)$, and $g(b) < f(b)$, then for some value c between a and b , $f(c) = g(c)$.
57. The range of a continuous function is connected if its domain of definition is connected.
58. If the graph of a bounded function is closed, the function is continuous.
59. If $f(x)$ and $g(x)$ are continuous functions on the same domain of definition D , then $f(x) \cdot g(x)$ is continuous on D .
60. If $f(x)$ and $g(x)$ are continuous functions on the same domain of definition D and 0 does not belong to the range of $g(x)$, then $f(x)/g(x)$ is continuous on D .
61. Any subset of a continuous graph is continuous.
62. If $f(x)$ ($a \leq x \leq b$) is continuous and $f(a) < N < f(b)$, there is a value c such that $f(c) = N$. In fact, there is a least such value c .

9.2 Exercises

1. Describe a function that is not continuous anywhere.
2. Describe a function that is continuous at only one point.
3. Describe a function $f(x)$ ($0 \leq x \leq 1$) that is continuous at the irrational values and discontinuous at the rational values.
4. Suppose $F(x, y)$ is a function defined over a plane domain. Give two definitions of what would be meant by $F(x, y)$ being continuous at a point of this domain, one corresponding to each of the definitions above.
5. If $f(x)$ ($0 < x < 1$) is continuous, does it assume its maximum?
6. Give an example of a bounded function whose graph is not bounded.

Chapter 10

Uniform Continuity

Definition 10.1 (Uniform continuity). The function $f(x)(x \in D)$ is ***uniformly continuous*** if for each positive number ε there is a positive number δ such that if x and y are numbers in D differing by less than δ , then $|f(x) - f(y)| < \varepsilon$.

We note that in the above definition the δ is not a function of any particular value x_0 of the domain of definition.

10.1 Exercises

1. Give an example of a function that is continuous but not uniformly continuous.
2. Give an example of a bounded function that is continuous but not uniformly continuous.

10.2 Theorems

Prove the following.

63. If G is an open covering of a sequentially compact set X , there is a positive number $\varepsilon(G)$ such that for each point x of X , $N(x, \varepsilon)$ lies in an element of G . (This number ε is called the Lebesgue number of the open covering G .)
64. A continuous function on a sequentially compact domain of definition is uniformly continuous.

Chapter 11

Sequences of Functions

Definition 11.1 (Convergence). A sequence of functions $\{f_1(x), f_2(x), \dots\}$ all defined on the same domain D is said to converge to a function $f(x)$ on D provided that for each value x of D , $\{f_1(x), f_2(x), \dots\}$ converges to $f(x)$.

Definition 11.2 (Equicontinuity). The sequence of functions

$$\{f_1(x), f_2(x), \dots\}$$

on D is said to be equicontinuous if for each positive number ε there is a positive number δ such that if x, y are values of D differing by less than δ , then for each integer i , $|f_i(x) - f_i(y)| < \varepsilon$.

11.1 Exercises

1. What is meant by saying that a collection of functions is equicontinuous?
2. Give an example of a sequence of uniformly continuous functions that converges to a function that is not continuous.

11.2 Theorems

Prove the following.

65. If a sequence of equicontinuous functions converges to a function $f(x)$, then $f(x)$ is uniformly continuous.
66. If $\{f_1(x), f_2(x), \dots\}$ is a bounded (there is a number N such that $-N < f_i(x) < N$) equicontinuous sequence of functions on the same domain

of definition, then some subsequence of $\{f_1(x), f_2(x), \dots\}$ converges to a continuous function on this domain.

Chapter 12

Slopes

Calculus is sometimes called a study of certain aspects of the topology of the real numbers. Functions are certain types of transformations, while derivatives and integrals are types of limits.

If $f(x)$ gets close to b as x gets near but not equal to a we say that $f(x)$ approaches b as x approaches a . More precisely, we give the following definition.

Definition 12.1 (Limit).

$$\lim_{x \rightarrow a} f(x) = b$$

means that a is a limit point of the domain D of f , and for each neighborhood N of b there is a deleted neighborhood U of a such that if $y \in D \cap U$, then $f(y) \in N$.

Definition 12.2 (Slope). Suppose G is the graph of a function, $M(x_p, x_q)$ denotes the slope of the secant line between $p = (x_p, f(x_p))$ and $q = (x_q, f(x_q))$, and p is a point of G which is a limit point of G . Then if

$$\lim_{x_q \rightarrow x_p} M(x_p, x_q)$$

exists, this limit is called the slope of G at p .

Definition 12.3 (Derivative). Suppose a is a point and a limit point of the domain of definition D of $f(x)$. If

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a},$$

then we call $f'(a)$ the **derivative** of f at a .

Disregarding the domain of definition of $f(x)$, as is often done in casual treatments of calculus, we have that

$$f'(a) = \lim_{\varepsilon \rightarrow 0} \frac{f(a + \varepsilon) - f(a)}{\varepsilon}$$

means that for each neighborhood N of $f'(a)$ there is a deleted neighborhood U of 0 such that $\frac{f(a+h)-f(a)}{h}$ is an element of N if h is in U .

Definition 12.4. A graph G is *rising* at the point p of G if there exist two vertical lines, one to the left and the other to the right of p such that every point of G between these vertical lines is higher or lower than p according as it is to the right or left of p .

12.1 Exercises

1. Give an ε, δ definition of what is meant by $\lim_{x \rightarrow a} f(x) = b$.
2. Give an ε, δ definition of what is meant by a function rising.
3. In the definition of limit, is it concluded that $f(a) = b$?
4. Make a definition of what would be meant by saying that $\lim_{x \rightarrow \infty} f(x) = b$.
5. Give an example of a function $f(x)$ ($0 \leq x \leq 1$) that has a derivative everywhere but whose derivative is not continuous at $x = 0$.
6. Give an ε, δ definition of a derivative.

12.2 Theorems

Prove the following.

67. If a function $f(x)$ ($0 \leq x \leq 1$) is rising at every point, it is continuous for at least one value.
68. If $f(x)$ ($a \leq x \leq b$) is rising at every point and $a \neq b$, then $f(a) < f(b)$.
69. If a graph G has a slope at a point, it is continuous there.
70. If the graph of $f(x)$ is rising at $(c, f(c))$, then $f'(c) > 0$ if $f'(c)$ exists.
71. If $f(x)$ ($a \leq x \leq b$) has a maximum at a value c between a and b then $f'(c) = 0$ if it exists.

72. If $f'(x)$ ($a \leq x \leq b$) exists and $a \neq b$, then there is a value c between a and b such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
73. If $f'(x)$ ($a \leq x \leq b$) exists and $f'(a) < N < f'(b)$, then there is a value c between a and b such that $f'(c) = N$.

Chapter 13

Transformations

Definition 13.1. A *transformation* f of X into Y is a function, rule, or law which assigns to each element x of X a unique element $f(x)$ of Y .

Definition 13.2. If for each point y of Y there is at least one element x of X such that $f(x) = y$, the transformation is said to take X *onto* Y .

Definition 13.3. If for each y of Y there is exactly one element x of X such that $f(x) = y$, the transformation is called *one-to-one* or (1-1).

Equations are examples of transformations of the real numbers into the real numbers. $f(x) = x^3$ is both (1-1) and onto, $f(x) = x^3 - x$ is onto but not (1-1), while $f(x) = x^2$ and $f(x) = \sin(x)$ are neither (1-1) nor onto.

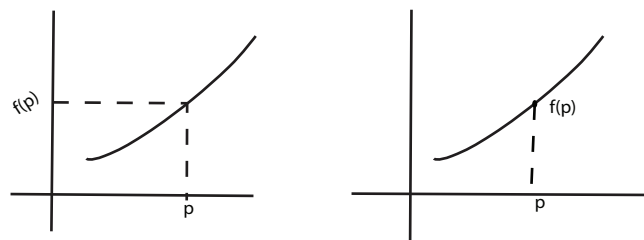
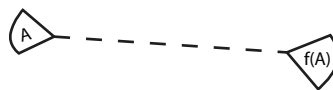
Rotations, translations, projections, congruences, and similarities are examples of transformations encountered in geometry.

Also, real valued functions may be regarded as transformations of points on the x -axis into points of the plane on the same vertical line with them. Hence $f(x) = x^2$ ($1 < x < 2$) may be regarded as a transformation of the part of the x -axis between $(1, 0)$ and $(2, 0)$ onto the part of the graph of $y = x^2$ between $(1, 1)$ and $(2, 4)$.

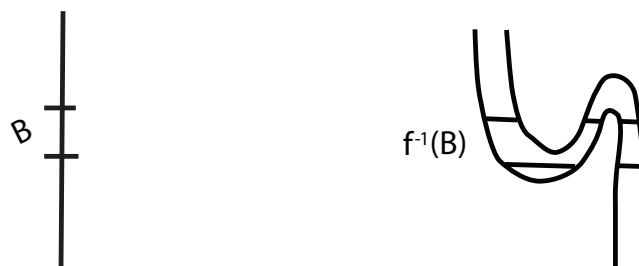
Furthermore, a real valued function may be regarded as a transformation on the x -axis. Then $f(x) = x^2$ would take the point $(a, 0)$ into the point $(0, a^2)$. See Figure 13.1.

Definition 13.4. If f is a transformation of X into Y and A is a subset of X , then the *image* of A , denoted $f(A)$, is the set $\{f(x) | x \in A\}$. See Figure 13.2 below.

Definition 13.5. If $B \subset f(X)$, the *inverse* of B , denoted $f^{-1}(B)$, is the set $\{x | f(x) \in B\}$.

Figure 13.1: Transformations on the x -axis.Figure 13.2: A and its image $f(A)$.

As an example of $f^{-1}(B)$, consider the example shown in Figure 13.3, in which the function f is a projection onto a line.

Figure 13.3: f is a Projection onto a line.

Definition 13.6 (Neighborhood definition of continuity). A transformation T of X into Y is **continuous at the point** x of X if for each neighborhood N of $f(x)$ there is a neighborhood $U(x, N)$ of x such that $T(U \cdot X) \subset N$. T is **continuous** if it is continuous at each point of X .

Definition 13.7. A continuous transformation is called a **mapping**.

Definition 13.8. The image of an interval under a mapping is called a **continuous curve**. (Also called a Peano curve.)

13.1 Exercises

1. Give an example of an into transformation that is not an onto transformation.

2. Describe a 1-1 transformation of the non-negative real numbers onto the reals.
3. Is there a transformation of the plane onto a point? Of a point onto the plane?
4. Give an ε, δ definition of continuity of a transformation.
5. Give a definition of uniform continuity of a transformation.
6. Show that a continuous transformation does not preserve any of the following properties individually:
 - (a) boundedness,
 - (b) being closed,
 - (c) being infinite,
 - (d) being unbounded,
 - (e) having area of 1 square foot,
 - (f) having length of 1 inch.
7. Suppose T is a continuous transformation of the plane into the plane. Which of the following properties implies that it is continuous at the origin?
 - (a) T is continuous on each line through the origin.
 - (b) T is continuous on each vertical and horizontal line.
 - (c) T is continuous on each straight line.
 - (d) T is equicontinuous on the lines through the origin.
 - (e) T is equicontinuous on all vertical lines and continuous on horizontal lines.

13.2 Theorems

Prove the following.

74. Connectedness is preserved by a continuous transformation.
75. Sequential compactness is preserved by a continuous transformation.
76. If T is a continuous transformation of X , then for each relatively open subset N of $T(X)$, $T^{-1}(N)$ is a relatively open subset of X .
77. A transformation is continuous if the inverse of open sets are open.

78. A square plus its interior is a continuous curve.
79. If x is a limit point of M and T is a continuous transformation of M , then $T(x)$ is either a point or a limit point of $T(M)$.
80. If $\{x_1, x_2, \dots\}$ converges to x_0 , then $\{T(x_1), T(x_2), \dots\}$ converges to $T(x_0)$ if T is a continuous transformation defined on a set containing the x 's.

Chapter 14

Function of a Function

If g is a map of X into Y and f is a map of Y into Z , then fg is a map of X into Z . To find $fg(x)$, first consider x , then its image $g(x)$ under g , and finally the image $fg(x)$ of $g(x)$ under f . See Figure 14.1.

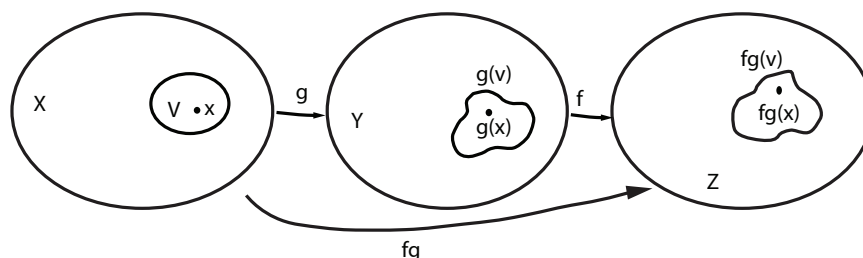


Figure 14.1: Function of a Function.

To show that fg is continuous at x , we need to show that for each neighborhood U of $fg(x)$ there is a neighborhood V of x such that $fg(V \cdot X) \subset U$. This is done by considering a neighborhood W of $g(x)$ such that $f(W \cdot Y) \subset U$ and then picking V so that $g(V \cdot X) \subset W$.

If f and g are real valued functions of real numbers, the graph of fg may be constructed from the graphs of f and g .

Here is how to plot $(a, fg(a))$. Using a , find the value of $g(a)$ on the graph of g . Using this value $g(a)$ thus obtained, find the value $f(g(a))$ on the graph of f . Then plot the point $(a, fg(a))$ on the graph fg .

In Figure 14.2, we let $g(x) = x^2 - 4$, $f(x) = \sin(x) - x$, and $fg(x) = \sin(x^2 - 4) - (x^2 - 4)$.

Of course in plotting, it is much better to get an idea of the general position of many points at once rather than plot them one at a time.

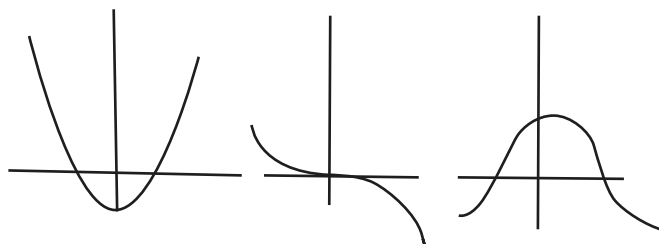


Figure 14.2: Several Graphs.

14.1 Theorem

Prove the following:

81. If g is a continuous transformation on X and f is a continuous transformation of $g(X)$, then fg is continuous on X .

Chapter 15

Homeomorphisms

Definition 15.1. If f is a (1-1) transformation of X onto Y , f^{-1} is a (1-1) transformation of Y onto X . If both f and f^{-1} are continuous, f is called a *homeomorphism*.

Not every (1-1) continuous transformation is a homeomorphism. For example, let $f(x)$ be the function defined for $x \geq 0$ by

$$f(x) = \left(\cos \left(\frac{2\pi x}{x+1} \right), \sin \left(\frac{2\pi x}{x+1} \right) \right).$$

$f(x)$ takes a ray continuously and (1-1) onto a circle in the plane, but f^{-1} is not continuous at $(1,0)$.

Definition 15.2. Two sets are *homeomorphic* or *topologically equivalent* if there is a homeomorphism of one onto the other.

Definition 15.3. Any set topologically equivalent to an interval is called an *arc*.

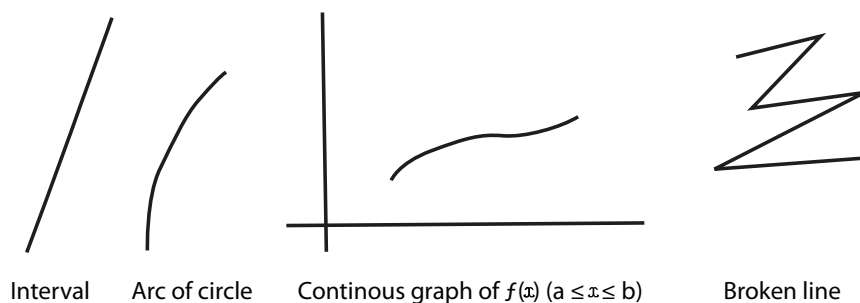


Figure 15.1: Some Arcs.

Definition 15.4. Any set topologically equivalent to a circle is called a *simple closed curve* or a *1-sphere*.

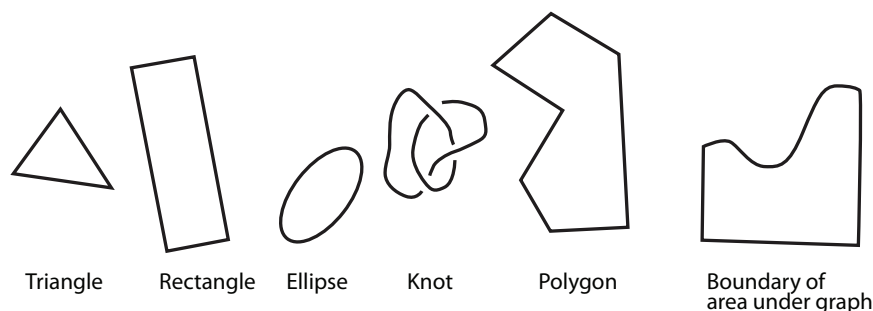


Figure 15.2: Some Simple Closed Curves.

It may be shown that in the plane the “interior” of any simple closed curve is topologically equivalent to the interior of a circle. This is part of the Jordan curve theorem.

When one speaks of the area of a circle, he probably refers to the area of its interior. There are simple closed curves that wiggle around so much that in a certain sense, they have no area. However, it is of importance in integral calculus that if G is the continuous graph of $y = f(x)$ ($a \leq x \leq b$) and $f(x) > 0$ and J is the simple closed curve which is the sum of G and the intervals from $(a, 0)$ to $(a, f(a))$, $(a, 0)$ to $(b, 0)$, and $(b, 0)$ to $(b, f(b))$, then the area of the interior of J does exist. It is called $\int_a^b f(x) dx$.

Definition 15.5. A set topologically equivalent to the surface of a sphere is called a *simple surface* or a *2-sphere*.

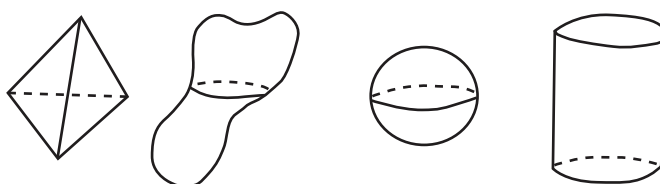


Figure 15.3: Some Simple Surfaces.

There are wild simple surfaces in Euclidean 3-space whose interiors are not topologically equivalent to the interior of a sphere.

Although two surfaces are topologically equivalent to each other it may not be possible to push, pull, and stretch one without breaking and tearing to make it fit on the other. See Figure 15.4 for a picture of two such surfaces.

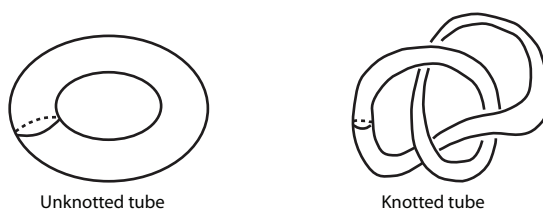


Figure 15.4: Topologically Equivalent, but . . .

Definition 15.6. A set is *nondegenerate* if it contains more than one element.

Definition 15.7. A *continuum* is a closed and connected point set.

15.1 Exercises

1. Describe a homeomorphism between a disk and a triangle plus its interior.
2. Show that a simple closed curve is bounded.
3. Show that an arc is not homeomorphic with a simple closed curve.
4. Let M be the sum of the graph of

$$y = \sin\left(\frac{1}{x}\right) \quad \left(-\frac{1}{\pi} \leq x \leq \frac{1}{\pi}, x \neq 0\right)$$

and the origin. Is M connected? Is M a continuum? Is there a (1-1) continuous transformation of M onto an interval?

15.2 Theorems

Prove the following.

82. A segment is homeomorphic with a line.
83. The plane is homeomorphic with the interior of a circle.
84. The surface of a sphere minus a point is homeomorphic with the plane.
85. An open subset of the plane is connected if and only if each pair of its points can be joined by an arc in it.

86. The homeomorphic image of a nondegenerate sequentially compact continuum is a nondegenerate sequentially compact continuum.
87. If p and q are two points of a simple closed curve J , then $J - (\{p\} + \{q\})$ is not connected.
88. If p is a non end point of an arc A , then $A - \{p\}$ is not connected.
89. There is an arc with no length.
90. The graph of a continuous function $f(x)$ ($a \leq x \leq b, a < b$) is an arc.

Chapter 16

Cantor Set

Definition 16.1. Let $M_1 = [0,1]$ be a straight line interval, $M_2 = M_1 - (1/3, 2/3)$ be the part of M_1 remaining after the open middle third of it is removed, $M_3 = M_2 - (1/9, 2/9) - (7/9, 8/9)$ be the part remaining after the open middle thirds of the intervals in M_2 are removed, \dots and so on. The intersection of M_1, M_2, \dots is called the **Cantor set** or the **Cantor discontinuum** or the **Cantor ternary set**.

Definition 16.2. A set is **perfect** if it is closed and every point of it is a limit point of it.

The Cantor set is an example of a perfect set that contains no nondegenerate continuum.

16.1 Exercises

1. Find the sum of the lengths removed in forming the Cantor set.
2. If at the first step the middle open third of $[0,1]$ was removed, at the next step the open middle ninths of the remaining intervals were removed, at the next step the open middle twenty sevenths were removed, \dots , show that the sum of the lengths removed is different from 1. Show that the resulting set is homeomorphic with a Cantor set.

16.2 Theorems

Prove the following.

-
91. An interval is the image of the Cantor set under some continuous transformation. (It can be shown that any sequentially compact set is such an image.)
 92. Any perfect set has uncountably many points.

Chapter 17

Baire property

Definition 17.1. A subset M of A is called *dense* in A if each point of A belongs to \overline{M} .

Definition 17.2. A set X is *separable* if some countable subset of X is dense in X .

Definition 17.3. A subset M of A is called *nowhere dense* in A if M does not contain any relatively open subset of A .

Definition 17.4. The intersection of a countable number of open sets is called an *inner limiting set* or a G_δ set.

Definition 17.5. The sum of a countable number of closed sets is called an F_σ set.

17.1 Theorems

Prove the following. Theorems 93 and 94 are extensions of Theorem 43.

93. The square plus its interior is not the sum of a countable number of nowhere dense sets.
94. (Baire property theorem.) No sequentially compact set X is the sum of a countable number of closed sets no one of which contains a relative open subset of X .
95. A closed set is an inner limiting set.
96. The complement of a G_δ set is an F_σ set and conversely.
97. No interval is the sum of a countable number more than one of mutually exclusive closed sets.

98. The plane is separable.

Notes to the Instructor

¹The restriction that the sets be closed is not needed here. If students don't notice it now, they will probably figure it out when they get to Theorem 8, one proof of which uses the stronger version of this theorem.

²Students will probably come up with a variety of possible properties here. One that might come up and would be particularly nice to discuss here is connectedness.

³My students (and I) always find this question confusing. I think the point is that we can include the empty set as a neighborhood if we want to. We can think of it as the interior of a circle with radius 0. Or, if we're thinking more abstractly, if we have a space satisfying Axioms 1 and 2 in which the empty set isn't a neighborhood, we can add it as a neighborhood without changing the resulting open sets.

⁴This definition was missing from the version of Bing's notes that was passed down and that I originally used. The version used here was interpolated from [2].

⁵In my experience, students often propose Lemmas and Corollaries while working on this section. We usually name these and incorporate them into the theorem sequence. Here are some examples taken from my classes:

1. If X_1 and X_2 are not mutually separated, $X_1 \subset A_1$, and $X_2 \subset A_2$, then A_1 and A_2 are not mutually separated.
2. If X is a connected set and A and B are non-null mutually separated sets such that $X \subset A + B$, then $X \subset A$ or $X \subset B$.
3. If A and B are mutually separated, then A is relatively open in $A + B$.
4. If A is the union of two disjoint closed sets, then it isn't connected.

⁶In Bing's original notes, he calls this property *compactness* and calls what we now call compactness *bicompactness*. This has been changed here to reflect modern usage.

Appendix A

Nathaniel Miller's Sample Midterm

I give an in-class midterm. I tell students ahead of time that the midterm will include several theorems and/or problems that have been presented in class, as well as some other (easier) problems. I like to include an extra credit problem for the stronger students. The following is the midterm I gave last time I taught the course. This midterm covered the materials in Chapters 1 and 2.

1. (20 points) Prove two of the following three Theorems, assuming only the preceding Theorems. (If you turn in all three, I will only grade the first two.)
 - (a) Theorem 6: The product of two open sets is open.
 - (b) Theorem 8: For each set X , \bar{X} is closed.
 - (c) Theorem 15: The complement of a closed set is open.
2. (20 points) Prove or disprove:
 - (a) Any (possibly infinite) union of closed sets is closed.
 - (b) Any (possibly infinite) intersection of closed sets is closed.

Definition T.1 A point p is an *interior point* of a set H iff there exists a neighborhood N_p of p such that $N_p \subset H$.

Definition T.2 The *interior* of a set S , denoted by $\text{INT}(S)$, consists of all points $p \in S$ that are interior points of S .

3. (20 points) Prove or disprove the following statement: for every set S , $\text{INT}(S)$ is the union of all of the open sets contained in S .

Definition T.3 If $f: A \rightarrow B$ is a function from A to B , and S is a subset of B , then the *inverse image* of S under f , denoted by $f^{-1}(S)$,

is defined to be the set of all elements $a \in A$ such that $f(a) \in S$. That is,

$$f^{-1}(S) = \{a \in A \mid f(a) \in S\}.$$

Definition T.4 Recall from calculus that a function $f: \mathbf{R} \rightarrow \mathbf{R}$ is defined to be *continuous* iff

$$(\forall x_0 \in \mathbf{R})(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbf{R})(|x - x_0| < \delta \rightarrow |f(x) - f(x_0)| < \varepsilon).$$

4. (Extra Credit) Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be given. Prove that the following are equivalent:

- (a) f is continuous.
- (b) If S is any open set in \mathbf{R} , then $f^{-1}(S)$ is also open.

Appendix B

Nathaniel Miller's Sample Final

I give a take home final in this class. I usually give it out at the next-to-last class, and it is due during our scheduled final exam meeting time, so students have about one week to complete it. This is the exam I gave last time I taught the class. Although it contains 12 problems, each student was only assigned 5 of them. I picked an appropriate selection of problems for each person, since some students were much stronger than other students. I have given this kind of take home exam several times, with different problems for different students, and I have never gotten any complaints. Students understand that different students are working at different levels. When this exam was given, we had studied Chapter 5 on sequential compactness but had not yet gotten to Chapter 7 on compactness.

Instructions

In completing this exam, you may consult your notes and handouts from this class, but you may not consult any other sources or discuss the problems with anyone other than me.

To receive full credit, your answers must be complete and correct, and written clearly so that one of your peers who was a reasonable skeptic could follow your arguments completely and be fully convinced by them.

You may assume any theorems previously proven in the packet, and you may assume any previous problem in working a later problem (even if you didn't do the earlier problem).

Your final copy will not be accepted if it does not include the following honor pledge written and signed:

"I pledge on my honor that this examination represents my own work in accordance with University and class rules; I have not consulted any outside

sources, nor have I given or received help in completing this exam.”

Definitions

Definition T.1 A point p is an *interior point* of a set H iff there exists a neighborhood N_p of p such that $N_p \subset H$.

Definition T.2 The *interior* of a set S , denoted by $\mathbf{INT}(S)$, consists of all points $p \in S$ that are interior points of S .

Definition T.3 If $f: A \rightarrow B$ is a function from A to B , and S is a subset of B , then the *inverse image* of S under f , denoted by $f^{-1}(S)$, is defined to be the set of all elements $a \in A$ such that $f(a) \in S$. That is,

$$f^{-1}(S) = \{a \in A \mid f(a) \in S\}.$$

Definition T.4 Recall from calculus that a function $f: \mathbf{R} \rightarrow \mathbf{R}$ is defined to be *continuous* iff

$$(\forall x_0 \in \mathbf{R})(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbf{R})(|x - x_0| < \delta \rightarrow |f(x) - f(x_0)| < \varepsilon).$$

Definition T.5 A function f is called a *homeomorphism* if f is one-to-one and onto, and both f and f^{-1} are continuous. If $f: A \rightarrow B$ is a homeomorphism, then A and B are said to be *homeomorphic* to one another.

Definition T.6 A set S is said to be *compact* if every open cover \mathcal{O} of S contains a finite subcover. An open cover of S is a collection of open sets such that every point $s \in S$ is in one of those open sets $O \in \mathcal{O}$.

Problems

Let $\mathbf{Bd}(S)$ denote the boundary of S .

1. Prove or disprove: $\mathbf{Bd}(S)$ is empty if and only if S is both open and closed.
2. Prove or disprove the following statement: $\mathbf{Bd}(S) = \overline{S} - \mathbf{INT}(S)$.
3. Prove: If A and B are mutually separated, C is connected, and $C \subset A \cup B$, then $C \subset A$ or $C \subset B$.
4. Prove or disprove: If C is a connected set, and \mathcal{G} is a (possibly infinite) collection of sets G_i such that no one of them is mutually separated from C , then the set $A = C \cup \bigcup_i G_i$ is connected.

5. Prove or disprove: \mathbf{R}^2 is connected.
6. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be given. Prove that the following are equivalent:
 - (a) f is continuous.
 - (b) If S is any open set in \mathbf{R} , then $f^{-1}(S)$ is also open.
7. Prove: The interval $(-1, 1)$ is homeomorphic to \mathbf{R} .
8. Prove: The open disc $\{p \mid d(p, (0, 0)) < 1\}$ is homeomorphic to \mathbf{R}^2 .
9. Prove or disprove: If A and B are homeomorphic, then A is connected if and only if B is.
10. Must a connected component of an open set in \mathbf{R}^2 be open? Prove your answer.
11. Prove or disprove: Every compact set is sequentially compact.
12. Prove or disprove: Every sequentially compact set is compact.

Bibliography

- [1] Parker, John, *R. L. Moore : Mathematician and Teacher*, Washington, DC: Mathematical Association of America, 2005.
- [2] Bing, R. H., “Elementary Point Set Topology,” Amer. Math. Monthly, Vol. 67, No. 7, part II, August-September, 1960.
- [3] Coppin, Charles A., W. Ted Mahavier, E. Lee May, and G. Edgar Parker, *The Moore Method: A Pathway to Student-Centered Instruction*, Washington, DC: Mathematical Association of America, 2009.