Teaching Writing and Proof-Writing Together

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Abstract

This article describes the synergy obtained by teaching proof-writing in the more general context of teaching how to write well. It describes a course, “Reasoning about Reasoning,” that was offered at Cornell University as a first-year writing seminar. The course was taught in an inquiry-based format, in which students worked on problems in class in groups and then wrote up their results as formal papers that went through a series of revisions.

Difficulty Level: High; Course Level: Non-traditional

1 Background and Context

This article describes a course, “Reasoning about Reasoning,” that was taught in the mathematics department at Cornell University as a first-year writing seminar. Cornell University is a private Ivy League university, and at the same time is the public land-grant institution of New York State. It is highly selective and has about 14,000 undergraduate students. Cornell has a writing-across-the-curriculum initiative in which first-year writing seminars are offered by many departments across the university. Most students take two such courses during their first year at Cornell, which they choose from the diverse collection of courses offered each semester. This course marked the first time that a first-year writing seminar was offered by the mathematics department, and it was developed and offered four times by the author as a graduate student.

Although first-year writing seminars are offered on many subjects and in many departments at Cornell, they all adhere to a set of common guidelines. Each course is organized around a sequence of six to ten formal essays that form a logical sequence. At least three of these essays are required to undergo a series of guided revisions. Half of the classroom time in each seminar is expected to be spent on work directly related to writing, and in order to allow for sustained interaction between the instructors and the students around their writing, each seminar is capped at 17 students. As part of the interaction, instructors are expected to meet in individual conferences with students at least twice during the semester. The goal of the writing program is to develop students’ abilities to write good analytical prose, and its philosophy is that this is a goal that is best achieved through a sustained experience writing within a particular discipline.

Because students get to pick the seminars in which they want to enroll through a lottery system, the students in this class came from a wide range of backgrounds. Some were intending to major in mathematics or other sciences and had exceptionally strong mathematical backgrounds, while others had picked the course on a whim because they thought the topic looked interesting, and were not intending to take any other math courses at all at Cornell. So, the course materials had to be flexible enough to accommodate many kinds of students.

Although writing and proof-writing are typically taught in different contexts, there are a lot of gains from teaching them together. The goal of most analytical writing classes is to help students acquire the skill of writing clear, convincing arguments. It is, of course, rare for a writing class to have mathematics as the subject of the writing, but mathematics is, in fact, remarkably well suited to the goal, since making clear and correct arguments is a central part of what mathematics is about. Furthermore, mathematics is virtually the only domain of inquiry in which there is wide agreement as to what constitutes a correct and coherent argument. Many writing classes focus on analyzing works of fiction, presumably because most writing classes are taught in English departments, by teachers whose expertise is in this area. However, it seems to me that mathematics is actually a better setting for teaching analytical writing because mathematical writing focuses on deductive reasoning, and there is therefore less ambiguity about what can be considered a correct argument.

At the same time, a writing seminar is a great setting for teaching mathematics in general, and proof-writing in particular. Students tend to enter mathematics classes, especially lower-level mathematics classes, expecting to be told facts and shown procedures to be mimicked. By contrast, they tend to enter a writing seminar expecting to think about some of the most important issues we would like them to think about in writing proofs: how to write clear and convincing arguments, and what makes an argument clear and convincing. Most mathematicians would agree that mathematics is much more about reasoning and making arguments than it is about memorizing procedures and facts. This is particularly true of proof-writing. So, in a strange way, a writing seminar might be an even better venue for teaching proof-writing than a mathematics class.

2 Description and implementation

2.1 The Structure of the Course

The mathematical content of the course, as I taught it, focused on examining how we know what is (mathematically) true. Students read excerpts from a variety of writings on the subject of reasoning, including Douglas Hofstadter's *Gödel, Escher, Bach* [9], Martin Gardner's *Logic Machines and Diagrams* [6], and Imre Lakatos's *Proofs and Refutations* [10]; writings that provided examples of mathematical reasoning, such as Euclid's *Elements* [5]; and writings that posed problems for students to reason about, such as Lewis Carroll's *A Tangled Tale* [2], David Henderson and Daina Taimiņa's *Experiencing Geometry* [7], and Raymond Smullyan's books *The Lady or the Tiger* [15] and *What is the Name of this Book?* [16]. Some of them were used as starting points for class discussions about reasoning that were independent of any specific assignment we were working on, while others, particularly those that posed problems, were used as the basis for written assignments.

The main focus of the class was on these papers that the students wrote. They formed a logical progression of interrelated topics that asked the students to reason about mathematical problems and to explain their reasoning, and then to reflect on and write about the process of mathematical reasoning. Several sample assignments are given in Appendix A. The problems that we looked at came mainly from logic and geometry, and particularly non-Euclidean geometry. These were fruitful areas to look at because they had relatively accessible problems that students were unlikely to have previously encountered.

A typical progression as we were working on a problem went as follows: first, I would pose a problem about an unfamiliar topic, and give the students some time to work on this problem in small groups in class. (For example, on the very first day of class, I would give out the Babylonian tablets assignment reproduced in Appendix A.1 and ask students to start working on it in groups, with no preamble or background beyond what is found in the assignment.) While the students were working in groups, I would move from group to group, discuss their progress in making sense of and solving the problem, and try to steer each group in a fruitful direction if they seemed to be stuck. Periodically, I would stop the groups and facilitate a whole-class discussion about the progress that the groups were making in order to share ideas between groups and to make any suggestions or comments that I wanted everyone to hear. If there were readings that were relevant to the problem, I would most often assign them to be read shortly after we had already started working on the problem in class, so that the students would get to start thinking about the problem with no preconceived ideas, and would hopefully then view the readings as support for what they were trying to understand rather than just as an assignment to be checked off a list. Once I felt that enough time had been spent working on the problem in class, I would assign a due date for a written first draft of the paper. After they had turned it in, I made comments on the papers, led a class discussion about common problems with the papers, and then gave students the opportunity to revise them one or two more times (depending on the assignment). I tried to make comments on the papers that would show students which parts were incorrect, incomplete, or unclear, without giving them specific instructions on how to revise them. As a result, many of my comments were phrased as questions.

The writing assignments all required students to write mathematical explanations. In some cases, these were proofs, while in other cases they were other kinds of explanations, such as how to carry out a procedure, why a definition made sense, or what reasoning the author had used to come to a conclusion. Within our class culture, proof was just one form of mathematical discourse we were trying to master, and our aims in writing proofs were the same as our aims in the other kinds of mathematical writing we were doing: to provide as clear and as convincing arguments as we could. This goal was emphasized again and again throughout the course: when we were discussing how to go about writing each paper; in the comments on each paper; and in our class discussions about how to revise them. I noted that coming up with convincing arguments about mathematical problems and communicating them clearly in writing is what professional mathematicians do, and is what makes someone a mathematician at any level. It sometimes took students a long time, a third of the course or more, to really believe that this was what I was expecting from them and that a big part of what I was grading them on was their ability to write an explanation that would be readable by, and convincing to, one of their peers. Once they understood this, however, the change in their writing and how they approached writing was dramatically noticeable.

2.2 Useful Lessons from Teaching Writing

There were a number of aspects of the course that were natural in the context of a writing seminar that turned out to be surprisingly useful in teaching proof-writing. One was that most of the assignments went through a series of revisions after I had a chance to make comments on them and discuss common problems with the class. This was a requirement of all writing seminars at Cornell, but I found that it made a big difference in how the students learned the material. Even when they didn't make many changes between drafts, the act of thinking about what they had written and how they could make it better really changed the way they approached the writing assignments in the first place, and emphasized that the goal of the class was to improve their reasoning and writing skills. Students wrote more carefully because they knew they were going to have to go back and provide justification for anything that was unclear. I think that this is the aspect of the course that made the biggest change in the students' mathematical writing, including proof writing, and the way they approached thinking about mathematics.

Another thing that we did was to pay close attention to the small details of how things were worded—in other words, to writing style. I think that this was also surprisingly helpful for the mathematical skills they were building. It has always been my experience that the more clearly a student can explain something, the better he or she understands it, and I found while teaching this class that this rule even extends to small details of sentence structure. As a mathematician with limited training in teaching writing, I found Joseph Williams's book *Style: Ten Lessons in Clarity and Grace* [17] to be extremely helpful in figuring out how to think about working with my students around issues of stylistic control and sentence structure. In the writing seminar, we spent a significant amount of class time looking at individual sentences from students' papers and discussing how to improve them. However, when I give writing assignments in regular mathematics classes, I spend much less time on this—not because it wouldn't be valuable, but because students are less open to doing this outside of a writing class.

A third aspect of the writing seminar format that I found beneficial in teaching proof-writing was the requirement that all students set up appointments to meet one-on-one with the instructor outside of class at least two times during the semester. This was a general requirement of the writing seminar program, but it turned out to be a valuable opportunity to make sure that all students were getting sufficient individual feedback, not just those students who chose to attend office hours.

Two other aspects of the course were slightly unusual, and worth mentioning because they seemed to be beneficial. One is that because we weren't following a textbook, I had each student sign up for one or two days during the semester when he or she would be responsible for taking notes on whatever happened in class that day, writing up them as html files, and then posting them on our class web page after I approved them. This worked out really well: it was a good experience for the students taking the notes, and we then had a record of what had happened in class each day which the students and I could refer back to later on. The notes often included summaries of the discussions of student proofs that had occurred in class. Another aspect of the course that seemed to be useful was that I set up a discussion board to which all students were required to post each week. Sometimes I gave a specific prompt, while other times they were just left to post about whatever aspects of the course they were currently thinking about. This led to some really nice online discussions outside of class time, which tended to be focused on areas of disagreement between students or places where they felt stuck on assignments.

3 Outcomes

In the end, this course was extremely successful. It was clear to me that the students' writing and reasoning skills significantly improved over the course of the semester. They had a better understanding of what a clear explanation required, were better able to judge for themselves if they had succeeded in giving one, and they were better at continuing to work on a problem until they understood it and could explain it well. Furthermore, student comments made it clear that they were aware of the changes. Here are several examples of student comments, taken from course evaluations and other writing that they did for the course, which demonstrate this awareness and also highlight some of the other strengths of a course like this:

- My writing has definitely improved—I am a much more focused and organized writer now ... and this is definitely attributable to my writing seminar. ... [The written comments on papers] were just enough to give me a hint at whatever I needed to do to improve a statement, but not too much so that my abstract thinking was inhibited.
- Thinking about revisions for the vertical angle theorem brings up lots of ideas on different ways to prove theorems. I did not realize I would have to be quite so detailed and specific. ... [S]eeing so many different approaches helped me realize there are different ways of thinking about the same concepts.
- One of the primary aspects of the course that I enjoyed was the instructor's constructive criticisms on our formal essays. Even though during the course my feelings toward these comments were anything but joyous, I really thought they helped me learn to write better. They taught me to make absolutely sure that I justified everything I wrote about extremely clearly.
- [I]t was very helpful that we could revise each paper multiple times. I believe my writing improves best this way. Overall, I would definitely recommend this course to future computer science and math majors, because it actually explains what it means to prove an argument, something which I never really understood until now.

The course also demonstrated to the math department and the university writing program that a mathematical first-year writing seminar was not only possible, but was a great venue for teaching both mathematics and writing skills. Although Cornell has one of the oldest writing-across-the-curriculum initiatives in the country, dating back to 1966, the math department had never offered a first-year writing seminar prior to this one. The first time I taught the course in 1999, it was as an experiment to see if it would be possible.

It turned out to be so successful that I was asked to teach it three more times before I graduated and left the university in 2001. Since then, Cornell has continued to offer first-year writing seminars in the math department on a regular basis, although their mathematical content has varied greatly depending on who was teaching them. Many other people have taught these seminars since I left, under titles such as "To Infinity and Beyond," "The Dementia of Dimension," "Pictures in 1000 words or less," "Experiencing Mathematics Through Writing," and "Certainty and Ambiguity: Exploring Mathematical Concepts Through Writing."

4 Extending the method

Teaching this writing seminar has also had a fairly significant impact on my subsequent teaching. Since leaving Cornell, I have been at the University of Northern Colorado, a much less selective school whose primary mission is preparing future teachers. When teaching math courses, I have continued to use many of the techniques described here for teaching proof-writing, and even some of the same assignments, such as using the assignments reproduced in Appendices A.2 and A.3 in geometry classes. I still often use the course structure described in Section 2.1 as the basis for many of my classes. I still routinely assign formal writing assignments that go through a series of revisions and are graded on the basis of their mathematical exposition in addition to their correctness, and I still feel that this is one of the most effective techniques I know for improving my students' mathematical communication and proof-writing skills. I have used the assignments in a wide variety of classes for undergraduate pre-service elementary and secondary teachers, Master's-level in-service teachers, and Ph.D. students in mathematics education. These classes have ranged in size from 7 to 37 students. However, the writing-intensive teaching methods are definitely best suited to classes of 15 to 25 students. For descriptions of some of the mathematics courses in which I have used these methods, see the detailed course notes for my Modern Geometry I [11], Modern Geometry II [12], and Mathematical Modeling [13] courses published in the online *Journal of Inquiry-Based Learning in Mathematics*. (Editors' note: also, for more information in this volume on the logistics of inquiry-based approaches in different courses, see Ellis-Monaghan [3], Ernst and Hodge [4], and Rault [14].) Thus, the ideas presented here certainly can be used outside of the setting of a writing seminar, and work well with a wide range of kinds of students.

References

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[4] Dana C. Ernst and Angie Hodge, "Within ϵ of Independence: An Attempt to Produce Independent Proof-Writers via IBL," in this volume.

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[8] H. V. Hilprecht, *The Babylonian Expedition of the University of Pennsylvania*, Series A: Cuneiform Texts, Vol. XX, Part 1: *Mathematical, Metrological and Chronological Tablets from the Temple Library of Nippur*, University of Pennsylvania Department of Archeology, Philadelphia, 1906.

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[10] Imre Lakatos, *Proofs and Refutations*, Cambridge University Press, Cambridge, England, 1976.

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[15] Raymond Smullyan, *The Lady or the Tiger*, Alfred A. Knopf, New York, 1986.

[16] Raymond Smullyan, *What is the Name of this Book?*, Prentice-Hall, Englewood Cliffs, NJ, 1978.

[17] Joseph M. Williams, *Style: Ten Lessons in Clarity and Grace*, Addison-Wesley Longman, New York, 2000.

Appendix

A Sample Writing Assignments

This appendix includes several sample formal writing assignments used. They are not all of the assignments we used, and, in any case, the assignments varied from semester to semester.

A.1 Babylonian Tablets

Figure 1 is a copy of the front (obverse) and back (reverse) of a clay tablet found by an archeological dig of an ancient Mesopotamian city.[1] The tablet probably dates from around 1350 BC. The symbols on the tablet are called cuneiform, i.e., wedge-shaped, because they are made up of single wedge-shaped marks that were impressed with a stylus upon the tablet while it was still wet.

Explain the meaning of the tablet. Draw as many conclusions as you can about its contents, but be sure to explain all of your reasoning as clearly as possible. You might find it helpful to imagine that you are writing an explanation to be placed in a museum with the tablet.

[1] I always gave this assignment to groups to work on on the very first day of class. The tablet, which, as students soon discover, shows the multiplication table for nines in the base 60 babylonian system, is from an archeological dig of the Temple Library of Nippur conducted by the University of Pennsylvania. This drawing of the tablet originally appeared in the subsequent report [8], and is reproduced with a very accessible commentary and explanation in *Episodes from Early Mathematics* [1].

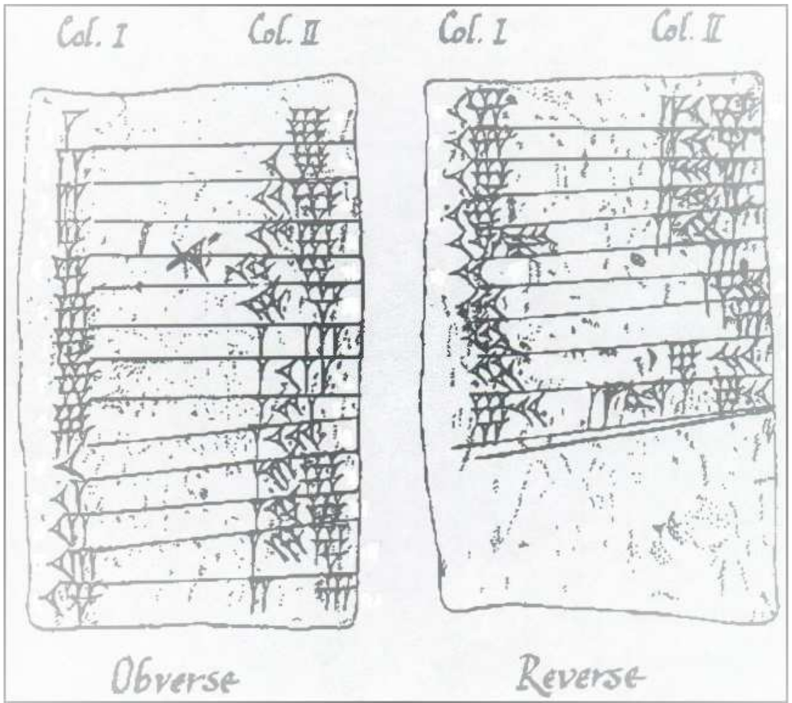

Figure 1: An ancient Babylonian tablet

A.2 Vertical Angle Theorem

For this assignment, you should define what you think an angle is, and use your definition to prove the vertical angle theorem.

In doing this, you can use Problems 3.1 and 3.2 from the reading[2] as a guide, but you need to integrate your answers to the questions into a unified, coherent whole. You should try to make your proof as clear and convincing as possible. The best proofs are those that not only convince the readers that something is true, but also allow them to understand why it is true. You should bear this in mind when you're deciding what assumptions to make and what kind of a proof to give. This is one reason that you want your assumptions to be simpler than the fact that you're trying to prove, and as obvious as possible. (It's also one reason that different people will give different proofs if they think of angles in different ways. Explaining why two angles have the same shape is different than explaining why they have the same degree measure.) If you decide to give a proof using degree measure, be careful not to confuse an angle (which is some kind of geometric object) with its degree measure (which is a number). If you want to talk about measuring angles, you'll need to explain how you want to assign a measurement, and also why angle measurements under your definition have any properties that you need for your proof. This is one reason that the authors of the reading suggest that proofs using symmetries are "generally simpler" than proofs involving measuring angles.

[2]The reading this refers to is an excerpt from *Experiencing Geometry* [7]. Problems 3.1 and 3.2, which appear in the excerpt along with quite a bit of supplementary explanation and suggestions of how to approach them, are as follows:

Problem 3.1: What is an Angle? Give some possible definitions of the term "angle." Do all of these definitions apply to the plane as well as to spheres? What are the advantages and disadvantages of each?

For each definition, what does it mean for two angles to be congruent? How can we check?

Problem 3.2: Vertical Angle Theorem (VAT) Prove: opposite angles formed by two intersecting straight lines are congruent. (Note: [Such] angles . . . are called **vertical angles**.) What properties of straight lines and/or the plane are you using in your proof? Does your proof also work on a sphere? Why? Which definitions from Problem 3.1 are you using in your proof?

A.3 Side-Angle-Side on a Sphere

Recall the following fact about planar triangles:[3]

(SAS) If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.

This is Euclid's Proposition 4.

Decide whether or not SAS is true on the sphere. If you think that it is true on the sphere, then give a convincing argument that shows that it is true. If you think that it isn't true, explain why it is false, show how to modify the statement to make it true on the sphere, and then prove your modified statement. In either case, explain and justify any decisions that you had to make in order to come to your conclusion. Try to write explanations that would convince a reasonable skeptic. You will probably find it helpful to thoroughly understand Euclid's planar proof before trying to decide what happens on the sphere.

A.4 Minesweeper

In a three-to-five page paper, explain what a strategy is and discuss the strategies that you developed to play the game Minesweeper.[4] Give a set of criteria for evaluating whether or not a given strategy is a good strategy. Try to be as concrete as possible, so that an independent observer could decide if a strategy meets your criteria with making any subjective decisions. Evaluate your strategies according to the criteria you developed. How are your criteria related to the kinds of reasoning that you were doing while you were playing the game?

A.5 Final Written Assignment in Lieu of a Final Exam

Complete both of the following:

A. Write an essay that connects two or more of the following topics discussed in class this semester: syntax vs. semantics; formal systems; the game of minesweeper; soundness and completeness; does .9999... = 1?; form vs. meaning in *Gödel, Escher, Bach*; the process of Proofs and Refutations; truth vs. provability; or any other topic discussed in class, subject to my approval.

B. We have read a lot of dialogues this semester.[5] Write your own 3–5 page dialogue illustrating ideas that we studied this semester that you think are particularly interesting. (Take this as an opportunity to show me something that you learned this semester.)

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[3]This assignment was given after we had read Euclid's Proposition 4, and also after we had done a lot of work deciding which lines are straight on the sphere and which of Euclid's postulates were true on the sphere. I still use this assignment along with the previous assignments in my Modern Geometry I class; for more details about how I use them, see my class notes in the *Journal of Inquiry-Based Learning in Mathematics* [11]. This assignment was also related to readings we did from Lakatos's *Proofs and Refutations* [10] about the role of finding counterexamples to refine conjectured theorems. The version of Proposition 4 given here is Sir Thomas Heath's literal translation of Euclid [5], and the language is a bit unconventional to modern ears; this is something we discussed in class.

[4]Prior to working on this, the students had been asked to spend time playing minesweeper while trying to be self-aware of the reasoning that they were doing. It relates directly to ideas of soundness and completeness of formal systems that we were reading about in *Gödel, Escher, Bach* [9].

[5]All of *Proofs and Refutations* [10] and large parts of *Gödel, Escher, Bach* [9] are written as dialogues.