Warning 1: Choose $n$ not $\Delta t$.

In order to multiply each velocity by $\Delta t$, we need to have the same size increment for each subinterval of time. This means we can only choose values of $\Delta t$ that divide evenly into the 2-hour time frame. For example, we could not use $\Delta t = 0.3$ hours, since six of these subintervals would get us 1.8 hours, but a seventh would put us over to the time 2.1 hours.

Instead, you can choose $n$ to be any positive integer you want. Then you can compute the corresponding $\Delta t$. Specifically, dividing the 2 hours into $n$ subintervals gives $\Delta t = \frac{2}{n}$.

In general, if you split an interval $[a,b]$ into $n$ equal subintervals, each will have size $\Delta t = \frac{b-a}{n}$.

Warning 2: Increase your ending value by $\Delta t/2$ on the calculator.

If your calculator rounds up the $\Delta t$ increment, it will fail to add the final term. This is because it starts at $t_0 = a$ for a left-hand sum (or $t_0 = a + \Delta t$ for a right-hand sum) and repeatedly increments by $\Delta t$ until it reaches $t_{n-1} = b - \Delta t$ (or $t_n = b$). If $\Delta t$ is rounded up even slightly, incrementing by these larger values will cause it to pass the ending point before it should and terminate the sum.

Using $b - \Delta t/2$ as the ending value for left-hand sums and $b + \Delta t/2$ for right-hand sums will solve this problem.

The Distance Traveled as a Definite Integral

In many cases, we can make our approximations as accurate as we wish by making $n$ sufficiently large (which corresponds to making $\Delta t$ sufficiently small). Expressed as a limit we would say that

$$x(2) = \lim_{n \to \infty} \sum_{i=1}^{n} v(t_i) \Delta t$$

The limit of this Riemann sum is called the definite integral of $v$ on the interval $[0,2]$, and is written

$$\int_{0}^{2} v(t) dt = \lim_{n \to \infty} \sum_{i=1}^{n} v(t_i) \Delta t.$$ 

The integral symbol $\int$ is meant to evoke an S for “sum” but is elongated to remind us that we are taking a limit of these sums as $n \to \infty$. The expression $v(t)dt$ is intended to evoke the idea of the product $v(t_i) \Delta t$. Of course in the limit, $\Delta t = 0$, so this isn’t a real interpretation, but it is helpful to remind us of the product structure.

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It is common to interpret the symbols
\[ \int_{0}^{2} v(t) \, dt \]
conceptually as meaning over the interval \( 0 \leq t \leq 2 \), we take an “infinite” sum of the “infinitesimal” products \( v(t) \, dt \), which are “infinitesimal” distances. The technical correction to this is that we are really taking the limit discussed above.

**The Definite Integral Generally**

This same structure occurs in a wide range of situations, and we define the definite integral of any function \( f \) over an interval \([a, b]\) in its domain by first considering partitioning the interval into \( n \) subintervals, each of size
\[ \Delta x = \frac{b-a}{n} . \]

The endpoints of the subinterval will then be\[ c_0 = a , \ c_1 = a + \Delta x , \ c_2 = a + 2\Delta x , \ldots , \ c_{n-1} = a + (n-1)\Delta x , \ c_n = a + n\Delta x = b . \]

We then choose points \( x_i \) in each of these subintervals where we will evaluate the function \( f \).

That is we choose these evaluation points so that
\[ c_{i-1} \leq x_i \leq c_i . \]

Then we define the definite integral as
\[ \int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x . \]

While it takes several steps to define this integral, these are exactly the steps you naturally took when approximating the unknown quantities in Activity 8.

**Different Choices for Evaluation Points \( x_i \)**

The freedom to choose the evaluation points \( x_i \) anywhere in each subinterval allows you to find underestimates and overestimates by evaluating \( f \) at the minimum or maximum on each subinterval. These are called **lower sums** and **upper sums** accordingly.

Simply choosing the leftmost or rightmost endpoints result in sums that we call **left-hand sums** and **right-hand sums**.
Graphical Interpretation

Definite integrals and the Riemann sums used to define them have a useful graphical interpretation. On a graph of the function $v(t)$ vs. $t$, the quantities that are multiplied together in each term of the Riemann sum are a height, $v(t)$, and a width, $\Delta t$. Thus the product is the area of a rectangle on the graph. These rectangles are illustrated below using the left-hand endpoints from our example with 12 subintervals.

The overestimate is represented by rectangles whose heights are determined by the velocity at the right hand side of each subinterval.
Making $n$ larger makes our approximations more accurate. The underestimate and overestimate for 100 subintervals are both shown on the graph below.

This image makes clear that the limit is the actual area bounded between the $t$-axis, the function, and the two vertical lines $t = 0$ and $t = 2$.

The errors for overestimates are whatever areas of the rectangles extend above the graph. The errors for the underestimates are whatever areas are not covered by the rectangle. These errors are shown in red below.
These errors can be bounded by the difference between the overestimate and underestimate, represented below by the red rectangles.