Labs 7-8: Hints on Linear and Quadratic Approximation

Here are some hints on how to approach the homework using linear and quadratic approximations. This will be in the context of approximating $\sqrt{3}$, but you should be able to translate these techniques directly to help with your homework.

Setting $f(x) = \sqrt{x}$, we know that $f'(4) = 2$. We’ll use the equation of a tangent line (linear approximation) and the equation of a “best fit parabola” (quadratic approximation) to start with $f(4) = 2$ and try to determine the decimal value of $f(3)$ without using a calculator.

We’ll compare results from using a linear approximation to the results from a quadratic approximation.

Equation of a line through the point $(x_0, y_0)$: $y = y_0 + m(x - x_0)$

Equation of a parabola through the point $(x_0, y_0)$: $y = y_0 + a(x - x_0) + b(x - x_0)^2$

Linear Approximation

For a linear approximation, we’ll need the first derivative:

$$f''(x) = \frac{1}{2\sqrt{x}} \quad \text{and} \quad f''(4) = \frac{1}{2\sqrt{4}} = 0.25.$$

The equation of the tangent line to $f$ at $(4,2)$ is

$$y = 2 + 0.25(x - 4).$$

Evaluating this at $x = 3$ gives us an approximate value for $f(3)$. That is

$$y(3) = 2 + 0.25(3 - 4) = 1.75.$$

See the graph on p. 3.
Quadratic Approximation

For a quadratic approximation, we’ll also need the second derivative:

\[ f''(x) = -\frac{1}{4} x^{-3/2} \text{ and } f''(4) = -\frac{1}{4} \cdot 4^{-3/2} = -0.03125. \]

The general equation of a parabola through \((4, 2)\) is

\[ y = 2 + a(x - 4) + b(x - 4)^2. \]

We just need to figure out what to make \(a\) and \(b\) in order to turn this into the “best fit quadratic” to \(f\) at the point \((4, 2)\). If we differentiate this equation, we get

\[ y'(x) = a + 2b(x - 4) \text{ and } y''(x) = 2b. \]

If we evaluate these at \(x = 4\), we get

\[ y'(4) = a \text{ and } y''(4) = 2b. \]

To make the first and second derivatives of this parabola match the first and second derivatives of \(f\) at the point \((4, 2)\), we can just set them equal to each other:

\[ y'(4) = f'(4) \text{ and } y''(4) = f''(4) \]
\[ a = 0.25 \text{ and } 2b = -0.03125 \]
\[ a = 0.25 \text{ and } b = -0.015625. \]

This makes the equation of the “best fit parabola” to \(f\) at the point \((4, 2)\)

\[ y = 2 + 0.25(x - 4) - 0.015625(x - 4)^2. \]

Now we can compute

\[ y(3) = 2 + 0.25(3 - 4) - 0.015625(3 - 4)^2 \]
\[ = 2 - 0.25 - 0.015625 \]
\[ = 1.73438 \]

(and note that a calculator was not needed).

See the graph on p. 3.
Linear Approximation

\[ y = 2 + 0.25(x - 4) \]

\[ f(x) = \sqrt{x} \]

Quadratic Approximation

\[ y = 2 + 0.25(x - 4) - 0.015625(x - 4)^2 \]

\[ f(x) = \sqrt{x} \]