Chapter 5 Exam

Name (print): ________________________________

Show all of your work.

1. The table below gives the speed, \( v(t) \) in meters per second, of a device dropped from a weather balloon for 80 seconds. Assume that the speed was increasing the entire time it fell.

<table>
<thead>
<tr>
<th>time ( t ) in s</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
</tr>
</thead>
<tbody>
<tr>
<td>speed ( v(t) ) in m/s</td>
<td>0</td>
<td>81</td>
<td>135</td>
<td>171</td>
<td>196</td>
<td>212</td>
<td>223</td>
<td>230</td>
<td>235</td>
</tr>
</tbody>
</table>

a. Using 4 equal subintervals, find an underestimate and overestimate for the total distance fallen.

Since the velocity is increasing the LHS will give an underestimate and the RHS will give an overestimate. For \( n = 4 \), \( \Delta t = 20 \).

Underestimate = LHS = \( 0 \cdot 20 + 135 \cdot 20 + 196 \cdot 20 + 223 \cdot 20 = 11,080 \) meters.

Overestimate = RHS = \( 135 \cdot 20 + 196 \cdot 20 + 223 \cdot 20 + 235 \cdot 20 = 15,780 \) barrels.

b. How frequently during the 80 second period would you need to measure \( v(t) \) to find an underestimate and overestimate for the distance that differ by 40 meters?

Since the velocity is increasing,

error < overestimate – underestimate = \( v(80)\Delta t - v(0)\Delta t = 235\Delta t \)

Setting \( 235\Delta t < 40 \), we get \( \Delta t < 0.170213 \). So we would need measurements at least every 0.170213 seconds.

Equivalently \( 235 \frac{80}{n} < 40 \), we get \( n > 470 \). So we would need at least 470 measurements.
2. a. Using 4 equal subdivisions, find a Riemann sum which is an overestimate for \( \int_{0}^{3} xe^{-x} \, dx \).

Since \( xe^{-x} \) is decreasing for \( x > 1 \), the LHS will be an overestimate. For \( n = 4 \), \( \Delta x = 0.5 \).

\[
LHS = 1e^{-1} \cdot 0.5 + 1.5e^{-1.5} \cdot 0.5 + 2e^{-2} \cdot 0.5 + 2.5e^{-2.5} \cdot 0.5 = 0.589229...
\]

b. Sketch a graphical representation of your Riemann sum, and write “LHS” or “RHS” next to your figure to indicate whether you are using a left-hand sum or a right-hand sum. Write out the terms of the Riemann sum using exact values (i.e., in terms of \( \ln \) and not decimal approximations from your calculator).

LHS = \( 1e^{-1} \cdot 0.5 + 1.5e^{-1.5} \cdot 0.5 + 2e^{-2} \cdot 0.5 + 2.5e^{-2.5} \cdot 0.5 = 0.59 \)

\[
\text{RHS} = 0.59
\]

c. Show that \( F(x) = -(x+1)e^{-x} + C \) is an antiderivative for \( xe^{-x} \), where \( C \) is any constant. In other words, show \( F'(x) = xe^{-x} \).

\[
F'(x) = -e^{-x} + (x+1)e^{-x} = xe^{-x}
\]

d. Using Part c, find the exact value of the integral \( \int_{0}^{3} xe^{-x} \, dx \).

\[
\int_{0}^{3} xe^{-x} \, dx = -4e^{-3} + 2e^{-1} = 0.53661...
\]
3. Find the area of the region bounded by the graph of \( y = x(x - 2)(x - 3) \) and the \( x \)-axis. Show all of the work required to compute this by hand.

The curve is above the \( x \)-axis from \( x = 0 \) to \( x = 2 \) and below the \( x \)-axis from \( x = 2 \) to \( x = 3 \). So the area bounded by the curve and the \( x \)-axis is

\[
\int_0^2 x(x - 2)(x - 3) \, dx - \int_2^3 x(x - 2)(x - 3) \, dx.
\]

An antiderivative of \( x(x - 2)(x - 3) = x^3 - 5x^2 + 6x \) is \( F(x) = \frac{1}{4}x^4 - \frac{5}{3}x^3 + 3x^2 \).

\[
\int_0^2 x(x - 2)(x - 3) \, dx = \left( \frac{4}{3} - \frac{40}{3} + 12 \right) = \frac{8}{3}.
\]

\[
\int_2^3 x(x - 2)(x - 3) \, dx = \left( \frac{9}{4} - 45 + 27 \right) - \left( \frac{9}{4} - \frac{40}{3} + 12 \right) = \frac{9}{4} - \frac{8}{3} = \frac{5}{12}.
\]

So the total area is \( \frac{8}{3} + \frac{5}{12} = \frac{37}{12} = 3.083 \).
4. Find the average value of \( f(x) = 3x^3 \) between \( x = 1 \) and \( x = 4 \). Show all of the work required to compute this by hand.

\[
\text{Average value} = \frac{1}{3} \int_1^4 3x^2 \, dx .
\]

An antiderivative is \( F(x) = x^3 \) so the average value is

\[
\frac{1}{3} \int_1^4 3x^2 \, dx = \frac{1}{3} [F(4) - F(1)] = \frac{1}{3} [64 - 1] = 21.
\]

5. For any positive integer \( n \), let \( \Delta x = \frac{3}{n} \) and

\[
x_0 = 2, \quad x_1 = 2 + \Delta x, \quad x_2 = 2 + 2\Delta x, \quad x_3 = 2 + 3\Delta x, \ldots, \quad x_n = 2 + n\Delta x .
\]

Find \( \lim_{n \to \infty} \sum_{i=1}^{n} 4x_i \Delta x \).

This is the limit of a (right-hand) Riemann sum for the function \( f(x) = 4x \).

The partition starts at \( x_0 = 2 \) and has length 3, so the interval is \([2,5]\).

Thus \( \lim_{n \to \infty} \sum_{i=1}^{n} 4x_i \Delta x = \int_2^5 4x \, dx \).

An antiderivative is \( F(x) = 2x^2 \), so \( \int_2^5 4x \, dx = F(5) - F(2) = 2 \cdot 5^2 - 2 \cdot 2^2 = 42 .\)