1. Find all of the critical points of $y = (x - 1)^3 (x + 3)^2$. Classify each one as a relative maximum, relative minimum, or neither and justify your answers.

$$y' = 3(x - 1)^2(x + 3)^2 + 2(x - 1)^3(x + 3)$$
$$= (x - 1)^2(x + 3)[3(x + 3) + 2(x - 1)]$$
$$= (x - 1)^2(x + 3)(5x + 7)$$

The critical points are at $x = -3$, $x = -7/5$, and $x = 1$.

$y'(-4) > 0$, $y'(-2) < 0$, $y'(0) > 0$, and $y'(2) > 0$.

So by the first derivative test,

$y$ has a relative maximum at $x = -3$,

$y$ has a relative minimum at $x = -7/5$,

and $y$ has neither a relative minimum or maximum at $x = 1$. 

2. Suppose \( f \) is a function whose derivative is \( f'(x) = \cos(x^2) \) and that \( f(0) = -2 \).

a. Does \( f \) have a critical point at \( x = 0 \)? Fully justify your answer.

No. Critical points are where the derivative is zero or undefined, and \( f'(0) = 1 \).

b. Does \( f \) have an inflection point at \( x = 0 \)? Fully justify your answer.

Yes. Inflection points are where the concavity changes.

\( f''(x) = -2x \sin(x^2) \), so when \(-\sqrt{\pi} < x < 0\), \( f''(x) > 0 \) so \( f \) is concave up.

when \(0 < x < \sqrt{\pi} \), \( f''(x) < 0 \) so \( f \) is concave down.

c. Find the equation of the tangent line to the graph of \( f \) at \( x = 0 \).

\( f'(0) = 1 \) so the equation of the tangent line is \( y = -2 + x \).

d. Use your answer to Part c to approximate \( f(0.5) \).

\( f(0.5) \approx -2 + 0.5 = -1.5 \)

e. Use the formula, \( \text{error} < \frac{M}{2} (x - x_0)^2 \)
to find an error bound for your approximation in Part d.

\[
\max_{0 \leq x \leq 0.5} \left| f''(x) \right| = 1 \quad \text{so error} < \frac{1}{2} \left( \frac{1}{2} \right)^2 = \frac{1}{8}.
\]

or

\[
\max_{0 \leq x \leq 0.5} \left| f''(x) \right| = \frac{1}{4} \quad \text{so error} < \frac{1}{8} \left( \frac{1}{2} \right)^2 = \frac{1}{32}.
\]
(30 points)

3. The equation \(3x^2 - 14x + 7y^2 + 2y - 7xy = 24\) is an ellipse passing through the three points \((0, -2)\), \((6, 0)\), and \((12, 6)\). Find the equations of the tangent lines to this ellipse at two of these points, showing all of your work.

Using implicit differentiation,

\[
6x - 14 + 14yy' + 2y' - 7y - 7xy' = 0
\]

\[
y' = \frac{7y - 6x + 14}{14y - 7x + 2}
\]

At \((0, -2)\), \(y' = 0\), so the equation of the tangent line is \(y = -2\).

At \((6, 0)\), \(y' = \frac{11}{20}\), so the equation of the tangent line is \(y = \frac{11}{20}(x - 6)\).

At \((12, 6)\), \(y' = -8\), so the equation of the tangent line is \(y = 6 - 8(x - 12)\).
(30 points)

4. If you have 100 meters of fencing and you want to enclose a rectangular area up against a long, straight wall, what is the largest area you can enclose? Show all of your work and explain how you know that your answer is a maximum.

\[ A = xy \]
\[ 2x + y = 100 \]
\[ A(x) = x(100 - 2x) = 100x - 2x^2 \]

with domain \( 0 \leq x \leq 50 \).

\[ A'(x) = 100 - 4x \]
\[ 0 = 100 - 4x \]
\[ x = 25 \]
\[ A'(10) > 0 \]
\[ A'(40) < 0 \]

so there is a maximum at \( x = 25 \) by the first derivative test.

The largest area is \( A(25) = 25(50) = 1250 \text{ m}^2 \).
(30 points)
5. Find the following limits, showing all of your work.

a. \[ \lim_{x \to 1} \frac{x - 1}{\sqrt[3]{x + 7} - 2} = \frac{0}{0} = \lim_{x \to 1} \frac{1}{\frac{1}{3}(x + 7)^{-2/3}} = 12 \]

b. \[ \lim_{x \to 0} \frac{e^{2x} - 2x - 1}{x^2} = \frac{0}{0} = \lim_{x \to 0} \frac{2e^{2x} - 2}{2x} = \frac{0}{0} = \lim_{x \to 0} \frac{4e^{2x}}{2} = 2 \]

c. \[ \lim_{x \to 0} \frac{3x^2 + 7x - 4}{2x^2 - x + 5} = \frac{-4}{5} \]