Problem 1. Many apartment complexes check your income and credit history before letting you rent an apartment. Let \( I = f(r) \) be the minimum annual gross income, in thousands of dollars, Apartments R Us requires of applicants who want to rent an apartment whose monthly rent is $r$.

(a) Assume that Apartments R Us requires an annual income of $20,000 to rent a $500 per month apartment and requires an annual income of $36,000 to rent a $1000 per month apartment. Assuming \( I \) is a linear function of \( r \), find a formula for \( I = f(r) \).

(b) Find \( f(800) \) and explain your answer in practical terms (say something about apartments, income, and rent!).

(c) Find \( f^{-1}(12) \) and explain your answer in practical terms (say something about apartments, income, and rent!).

Solution.

(a) Using point-slope form, we can write \( I = f(r) = 20 + \frac{16}{500} (r - 500) \). Simplifying, this we obtain \( I = f(r) = 4 + \frac{16}{500} r \).

(b) In order to rent an $800 apartment, Apartments R Us requires an annual income of 29.6 thousand dollars.

(c) Someone with an annual income of 12 thousand dollars can rent an apartment that costs $250 per month from Apartments R Us.

Problem 2. The population of Nicaragua was 3.6 million at the beginning of 1990 and growing at an annual rate of 3.4% per year. Let \( t \) be time in years since 1990.

(a) Express \( P \) as a function of \( t \).

(b) What is the doubling time of the population?

(c) How much did the population increase between the beginning and end of 2007?

(d) If \( P = f(t) \) is the function you found in (a), find a formula for \( f^{-1} \).

(e) Explain the practical meaning of \( f^{-1} \). Include units and be careful to say what you mean and mean what you say.

(f) Find \( f^{-1}(15) \) and explain the practical meaning of your computation. Pay attention to units!

Solution.

(a) \( P(t) = 3.6(1.034)^t \)

(b) \( \ln(2)/\ln(1.034) \approx 20.7 \) years

(c) \( P(18) - P(17) = 3.6(1.034)^{18} - 3.6(1.034)^{17} \approx 216,000 \) people

(d) \( t = f^{-1}(P) = \ln(P/3.6)/\ln(1.034) \)

(e) Given the population in millions, \( f^{-1}(P) \) says when, in years since 1990, the population of Nicaragua had or will have population \( P \), in millions.

(f) \( f^{-1}(15) = \ln(15/3.6)/\ln(1.034) \approx 42.7 \), which means the model predicts the population will be 15 million about 42.7 years after 1990, or in late 2032.
Problem 3. Let \( f(x) = \begin{cases} \frac{x^2 - 2}{2}, & 0 < x < 3 \\ 2, & x = 3 \\ 2x + 1, & 3 < x \end{cases} \)

(a) Compute the following limits:
\[
\lim_{x \to 3^-} f(x), \quad \lim_{x \to 3^+} f(x), \quad \lim_{x \to 3} f(x)
\]

(b) Is \( f \) continuous at \( x = 3 \)? Explain why or why not. If not, categorize the type of discontinuity.

Solution.

(a) All three limits are 7.

(b) \( f \) is not continuous at \( x = 3 \) since \( \lim_{x \to 3^-} f(x) = 7 \) is not equal to \( f(3) = 2 \). Since the limit exists, this is a removable discontinuity.

Problem 4. Solve for \( x \) using logs: \( 4 \cdot 3^x = 7 \cdot 5^x \). Give an exact answer (not an approximation).

Solution. \( x = \ln(4/7)/\ln(5/3) \)

Problem 5: Consider the function \( f \) given by the following graph of \( y = f(x) \):

(a) Find the domain and range of \( f \).

(b) Find all the points of discontinuity of \( f(x) \) and classify them (as jump, removable, or infinite).
(c) Find the following quantities (if they exist):

\[ \lim_{x \to 4^-} f(x) \quad \lim_{x \to 4^+} f(x) \quad \lim_{x \to 4} f(x) \quad f(-4) \]

\[ \lim_{x \to 1^-} f(x) \quad \lim_{x \to 1^+} f(x) \quad \lim_{x \to 1} f(x) \quad f(-1) \]

\[ \lim_{x \to 1^-} f(x) \quad \lim_{x \to 1^+} f(x) \quad \lim_{x \to 1} f(x) \quad f(1) \]

\[ \lim_{x \to 2^-} f(x) \quad \lim_{x \to 2^+} f(x) \quad \lim_{x \to 2} f(x) \quad f(2) \]

\[ \lim_{x \to 7^-} f(x) \quad \lim_{x \to 7^+} f(x) \quad \lim_{x \to 7} f(x) \quad f(7) \]

Solution.

(a) The domain is \([-4,6) \cup (6,7) \cup (7,8]\). The range is \((\infty,3]\cup\{5\}\).

(b) The function \(f\) has removable discontinuities at \(x = -1\) and \(x = 6\),
a jump discontinuity at \(x = 2\),
and a vertical asymptote at \(x = 7\).
In addition, the function is only left-continuous at \(x = -4\) and right continuous at \(x = 8\).

(c) Since \(f\) is not defined to the left of \(x = -4\), \(\lim_{x \to 4^-} f(x)\) and \(\lim_{x \to 4^+} f(x)\) do not exist, however
\[ \lim_{x \to 4^-} f(x) = f(-4) = 3 \]
\[ \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = 1, \text{ but } f(-1) = 5 \]
We can see that \(f\) is continuous at \(x = 1\), so \(\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = f(1) = -1\).
At the point \(x = 2\), \(\lim_{x \to 2^-} f(x) = f(2) = -2\), \(\lim_{x \to 2^+} f(x) = 1\), and since the left and right limits are not equal, \(\lim_{x \to 2} f(x)\) does not exist.
The function has a vertical asymptote at \(x = 7\), so \(\lim_{x \to 7^-} f(x) = \lim_{x \to 7^+} f(x) = \lim_{x \to 7} f(x) = -\infty\), and \(f(7)\) is not defined.

Problem 6. Write a formula representing the energy \(E\), expended by a dolphin swimming for 5 minutes given that it is proportional to the cube of the speed, \(v\), of the dolphin.

Solution. \(E = kv^3\), where \(k\) is some constant.
Problem 7. The point $P$ is rotating around a circle of radius 5 shown in the figure below. The angle $\theta$, in radians, is given as a function of time, $t$, by the graph.

(a) Estimate the coordinates of $P$ when $t = 1$.

(b) Estimate the time(s) at which $P$ touches or crosses each axis.

(c) Describe in words the motion of the point $P$ on the circle.

(d) Sketch a graph of the $x$ coordinate of $P$ as a function of time. Number and label the axes.

Solution.

(a) Looking at the graph of $\theta$ vs. $t$, we see that $\theta(1) \approx 4$ radians, so point $P$ will have traveled 4 out of the $2\pi$ radians around the circle (63.662% of the way). So $P$ will be in the 3$^{rd}$ quadrant and a reasonable guess might be $x \approx -3$ and $y \approx -4$. Although the question does not ask for a precise calculation, the coordinates will actually be $(5 \cdot \cos 4, 5 \cdot \sin 4) \approx (-3.3, -3.8)$.

(b) Point $P$ crosses the $x$-axis when $\theta = 0$, $\pi$, or $2\pi$. From the graph of $\theta$ vs. $t$, we can see this happens when $t \approx 0, 0.75, 2.5, 4.25,$ and $5$.

Point $P$ crosses the $y$-axis when $\theta = \pi/2$, or $3\pi/2$. From the graph of $\theta$ vs. $t$, we can see this happens when $t \approx 0.3, 1.2, 3.8,$ and $4.7$.

(c) $P$ starts at the right side of the circle, at point $(5,0)$ and starts moving counter-clockwise quickly (the graph is initially steep). It slows down as it goes around the circle in 2.5 units of time, eventually coming to a momentary stop (the graph is flat there) when it makes it one full revolution, and is back at $(5,0)$. Then the point turns around and slowly starts to travel clockwise, speeding up until it makes one revolution and is once again at point $(5,0)$.
Problem 8. Consider the functions in the figure below. Find the coordinates of \( C \). Show all of your work.

![Graph](image)

Solution. The equation of the line is \( y = 3 - 2x \). Setting this equal to the equation of the parabola, we get \( x^2 = 3 - 2x \) or \( x^2 + 2x - 3 = 0 \) which factors to \( (x+3)(x-1) = 0 \). So the intersection points are when \( x = -3 \) and \( x = 1 \) (the latter being the one already provided in the graph). Plugging in \( x = -3 \) to either equation to get the \( y \)-coordinate of \( C \) gives \( y = 9 \). Thus \( C = (-3,9) \).

Problem 9. You drive at a constant speed from Denver to Cheyenne, a distance of about 100 miles. About 60 miles from Denver you pass by Fort Collins. Sketch a graph of your distance from Fort Collins as a function of time.

Solution. The length of time and units are not provided, so we’ll assume the trip took a total time of \( T \) hours.

![Graph](image)
Problem 10. A cup of coffee contains 100 mg of caffeine, which leaves the body at a continuous rate of 17% per hour. (Note the word continuous!)

(a) Write a formula for the amount $A$, in mg, of caffeine in the body $t$ hours after drinking a cup of coffee.

(b) What is the half-life of caffeine in the body?

(c) What is the instantaneous rate of change at which caffeine leaves the body when there are 20 mg remaining?

(d) What is the actual percent change in caffeine in the body during a period of one hour?

(e) How much caffeine is left in the body 3 hours after drinking a cup?

(f) What is the average rate of change at which caffeine leaves the body in the first 3 hours after drinking a cup?

(g) Explain the meaning of the average rate you found in Part (f).

Solution.

(a) We are given the initial amount $A = 100$ mg and the continuous rate $k = -0.17$, so we can write down the equation using base $e$, as $A(t) = 100e^{-0.17t}$.

(b) We can solve for the amount of time it takes to decay from 100 mg to 50 mg by setting $A(t) = 50$ then solving for $t$. Specifically,

$$50 = 100e^{-0.17t}$$

$$\frac{1}{2} = e^{-0.17t}$$

$$\ln\left(\frac{1}{2}\right) = -0.17t$$

$$t = \frac{-\ln\left(\frac{1}{2}\right)}{-0.17} \approx 4.077336$$

So the half-life of caffeine in the body is about 4.08 hours.

(c) Since the caffeine is leaving the body at a continuous rate of $k=17\%$ per hour, when there is 20 mg, the instantaneous rate of change will be $20(-0.17) = -3.4$ mg per hour.

(d) In one hour, the amount of caffeine is decreased by a factor of $e^{-0.17} \approx 0.843665$. So the actual percent change is about $15.63\%$ decrease per hour.

(e) Three hours after drinking a cup of coffee, there will be $A(3) = 100e^{-0.17\times3} \approx 60.05$ mg remaining.

(f) The average rate of change during these first 3 hours is $\frac{\Delta A}{\Delta t} = \frac{60.05 - 100}{3 - 0} \approx -13.31$ mg/hr.

(g) The meaning of this average rate is that the caffeine would have to leave the body at a constant rate of 13.31 mg per hour in order to have the same change in the amount of caffeine ($-39.95$ mg) in the same amount of time (3 hours).
Problem 11. Find the equation of a line through the point \( \left( \pi/6, \sqrt{3}/2 \right) \) with slope \(-1/2\).

Solution. Using the point-slope form of a line, we can write down \( y = \frac{\sqrt{3}}{2} - \frac{1}{2} \left( x - \frac{\pi}{6} \right) \).

Problem 12. Sketch a graph of the height of water in the bottle shown below as a function of the volume of the water. Explain your reasoning by referring to only amounts of change in volume and height (without talking about rate of change).

Solution. The graph of \( h \) vs. \( V \) will always be increasing since adding more water will cause the height to go up. Furthermore, when the bottle is narrow, a fixed amount of water, \( \Delta V \), will cause a greater change in height, \( \Delta h \), than that same change in volume would cause when the bottle is wide. Graphically, a bigger \( \Delta h \) (rise) for the same \( \Delta V \) (run) will cause the slope to be steeper. A smaller \( \Delta h \) (rise) for the same \( \Delta V \) (run) will cause the slope to be less steep. Thus the slope of the graph of \( h \) vs. \( V \) will start off steep (corresponding to the bottom of the bottle), flatten out (widest part of the bottle), become steeper (approaching the neck), then increase at a constant rate (the neck of the bottle).

Problem 13. Let \( P \) be the function given by \( P(t) = 7.5 \cdot 6^t \).

(a) Rewrite \( P \) in the form \( P(t) = P_0 \cdot e^{kt} \) (that is, start by finding \( P_0 \) and \( k \)).

(b) Since \( P \) is an exponential function, its instantaneous rate of change \( \frac{dP}{dt} \) is proportional to the amount \( P \). What is the constant of proportionality?

(c) Find the instantaneous rate of change \( \frac{dP}{dt} \) when \( P(t) = 37 \) (without taking any derivatives – a later topic in the course).

(d) What is the doubling time for \( P \)?
Solution.

(a) To rewrite \( P \) as requested, we need to change the base, \( 6 = e^k \). So \( k = \ln 6 \). Thus
\[
P(t) = 7.5 \cdot e^{(\ln 6)t}
\]

(b) \( k = \ln 6 \)

(c) Generally, \( \frac{dp}{dt} = (\ln 6)P(t) \). So when \( P(t) = 37 \), the instantaneous rate is \( \frac{dp}{dt} = (\ln 6) \cdot 37 \approx 66.3 \)

(d) The doubling time is the value of \( t \) so that the factor \( 6^t = 2 \). Thus \( t = \ln 2/\ln 6 \approx 0.3869 \) units of time.

Problem 14. Consider the function \( f(x) = \frac{5^x - 1}{x} \).

(a) Fill in the following table of values for \( f(x) \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>–0.1</th>
<th>–0.01</th>
<th>–0.001</th>
<th>–0.0001</th>
<th>0.0001</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>1.4866</td>
<td>1.5966</td>
<td>1.6081</td>
<td>1.6093</td>
<td>1.6096</td>
<td>1.6107</td>
<td>1.6225</td>
<td>1.7462</td>
</tr>
</tbody>
</table>

(b) Based on your table of values, what would you expect the limit of \( f(x) \) as \( x \) approaches zero to be?
\[
\lim_{x \to 0} \frac{5^x - 1}{x} = \]

(c) Graph the function to see if it is consistent with your answers to parts (a) and (b). By graphing, find an interval for \( x \) near zero such that the difference between your conjectured limit and the value of the function is less than 0.01. In other words, find a window of height 0.02 such that the graph exits the sides of the window and not the top or bottom. What is the window?

Solution.

(a) The table values are

<table>
<thead>
<tr>
<th>( x )</th>
<th>–0.1</th>
<th>–0.01</th>
<th>–0.001</th>
<th>–0.0001</th>
<th>0.0001</th>
<th>0.001</th>
<th>0.01</th>
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<td>1.6096</td>
<td>1.6107</td>
<td>1.6225</td>
<td>1.7462</td>
</tr>
</tbody>
</table>

(b) \( \lim_{x \to 0} \frac{5^x - 1}{x} \approx 1.60945 \)

(c) Since \( 1.6107 - 1.5966 = 0.0141 \), we can use a window with
\[
-0.01 \leq x \leq 0.01 \\
1.5966 \leq y \leq 1.6107
\]
Problem 15. What are the ranges of the sine and cosine functions if we restrict their domain to the interval \([\alpha, \beta]\) where \(\alpha\) and \(\beta\) are the angle measures indicated in the diagram below? In other words, what output values can \(\sin \theta\) and \(\cos \theta\) produce if the input values of \(\theta\) are restricted so that \(\alpha \leq \theta \leq \beta\)?

\[\alpha = (0, 0), \beta = (0, 1), (0.39, 0.92), (-0.8, 0.6)\]

Solution. Since \(\cos \theta\) is the \(x\)-coordinate of the point on the unit circle intersected at angle \(\theta\), we can read off that the possible values for \(\cos \theta\) are \(-0.8 \leq x \leq 0.39\). Since \(\sin \theta\) is the \(y\)-coordinate of the point on the unit circle intersected at angle \(\theta\), we can read off that the possible values for \(\sin \theta\) are \(0.6 \leq y \leq 1\).

Problem 16. It takes me 15 minutes to drive 8 miles to work. What is my average speed during this trip (in miles per hour)? What is the meaning of that average speed?

Solution. The average speed is

\[
\frac{\Delta d}{\Delta t} = \frac{8 \text{ miles}}{0.25 \text{ hours}} = 32 \text{ miles per hour.}
\]

This means that, in order to drive the same distance (8 miles) in the same amount of time (15 minutes) at a constant speed, you would have to drive at 32 miles per hour.

Problem 17. For the functions \(f(x) = 3e^x\) and \(g(x) = x^2\), find each of the following:

(a) \(f(g(1))\)
(b) \(g(f(1))\)
(c) \(f(g(x))\)
(d) \(g(f(x))\)
(e) \(f(x)g(x)\)
Solution.

(a) \( f(g(1)) = f(1^2) = f(1) = 3e^1 = 3e \)

(b) \( g(f(1)) = g(3e^1) = g(3e) = (3e)^2 = 9e^2 \)

(c) \( f(g(x)) = f(x^2) = 3e^{x^2} \)

(d) \( g(f(x)) = g(3e^x) = (3e^x)^2 = 9e^{2x} \)

(e) \( f(x)g(x) = (3e^x)(x^2) = 3x^2e^x \)

Problem 18. Write the function \( h(x) = e^{x^3 + 3x + 10} \) as the composition of two functions, \( f(x) \) and \( g(x) \), where one function is exponential and the other is a polynomial.

Solution. If \( f(x) = e^x \) and \( g(x) = x^3 + 3x + 10 \), then \( h(x) = f(g(x)) \).

Problem 19. A cubic polynomial with positive leading coefficient is shown for \(-10 \leq x \leq 10\) and \(-10 \leq y \leq 10\) in the graph below.

(a) In total, how many zeros does this function have?

(b) Indicate the number of zeros the function has in each of the following intervals:

(i) \(-\infty < x < -10:\)

(ii) \(-10 < x < -5:\)

(iii) \(-5 < x < 0:\)

(iv) \(0 < x < 5:\)

(v) \(5 < x < 10:\)

(vi) \(10 < x < \infty:\)
Solution. The function is a cubic polynomial with positive leading coefficient, so as $x$ approaches $-\infty$, $f(x)$ must approach $-\infty$, and similarly as $x$ approaches $+\infty$, $f(x)$ must approach $+\infty$. Since the figure given above shows that the function turns around once, we know that the function has the shape shown below. Therefore, we can see that there must be 3 zeros, one each for $-5 < x < 0$, $0 < x < 5$ and one for $-\infty < x < 10$.

Problem 20. Below is a graph of the decreasing function $f$ defined by $f(x) = (1 + 2x)^{1/3}$. Let $h = \lim_{x \to 0} f(x)$. Without any fancy limit calculations, we can only approximate the value of $h$. For what values of $x$ can you be sure that $f(x)$ will approximate $h$ to within an accuracy of 0.1? Explain your reasoning.

Solution. Using $x = -0.1$ and $x = 0.1$, we get approximations for $h$ of $f(-0.1) = 0.8^{-10} = 9.3132$ and $f(0.1) = 1.2^{10} = 6.1917$. Using these as an overestimate and underestimate, respectively, we can only be sure that the error is less than $9.3132 - 6.1917 = 3.1215$. So we cannot say these are within 0.1 of the actual value of $h$.

Using $x = -0.001$ and $x = 0.001$, we get approximations for $h$ of $f(-0.001) = 0.998^{-1000} = 7.4039$ and $f(0.001) = 1.002^{1000} = 7.3743$. Using these as an overestimate and underestimate, respectively, we can be sure that the error is less than $7.4039 - 7.3743 = 0.0296$ which is less than the required 0.1 error bound.