Penny Moves Closure Activity – Lesson Plan

Lesson Overview
In this activity, closure is explored via spatial movement rather, thus expanding the notion of closure to other mathematical contexts. The movements here are defined by how a penny can move in a 2x2 grid. The operation used on two such movements is called composition. So, the question of closure is considered by asking whether one legal penny move followed by (composed with) a second legal penny move results in one of the defined legal penny moves. This activity can also be used to relate these special movements with the properties of identity, inverse, and commutativity in addition and multiplication.

Lesson Materials
- One penny or other marker for each student.
- A copy of the Penny Moves Closure Activity worksheet for each student.
- Optional: an overhead project and transparency to model the penny moves.

Lesson Introduction – Whole Class
1. Review the definition of closure. Key points:
   - To determine closure, a set of objects and an operation are needed.
   - An operation is closed on a set if each possible operation on two objects in the set results in an object in the set.
   - Sometimes it is helpful to organize your work to keep track of all the possible combinations. Tables are often useful when dealing with a finite set of objects.

2. Define the set of objects used in this activity. These objects are a set of moves a penny can make in a 2x2 grid. Each of these moves is denoted by a letter.
   I  The penny stays stationary.
   H  The penny moves horizontally (left to right or right to left) such that it remains in the grid.
   V  The penny moves vertically (top to bottom or bottom to top) such that it remains in the grid.
   D  The penny moves diagonally (down diagonally or up diagonally) such that it remains in the grid.

   Examples:
   a) If a penny is in square 3, an H move moves the penny to square 4.
   b) If a penny is in square 4, an H move moves the penny to square 3.
   c) If a penny is in square 2, a D move moves the penny to square 3.
   d) If a penny is in square 2, an I move leaves the penny in square 2.

   Have the students try:
   a) If a penny is in square 1, into what square does H move the penny? Answer: 2
   b) If a penny is in square 2, into what square does D move the penny? Answer: 3
c) If a penny is in square 3, into what square does D move the penny? Answer: 2

3. Define the operation. The name of the operation is composition. The symbol for composition is \( \circ \). So, \( H \circ V \) is said as, “H composed with V”. This means, do the H move first, then do the V move.

Examples:
   a. If a penny is in square 1, then \( H \circ V \) means first move the penny to square 2 and then to square 4.
   b. If a penny is in square 4, then \( I \circ D \) means first leave the penny in square 4 and then move it to square 1.

Have the students try:
   a) If a penny is in square 1, into what square does \( H \circ H \) move the penny? Answer: 1
   b) If a penny is in square 2, into what square does \( D \circ H \) move the penny? Answer: 4
   c) If a penny is in square 3, into what square does \( H \circ D \) move the penny? Answer: 1

4. Describe how to determine what move the composition of two moves results in. To find the answer to the composition of two moves, you find what one move would move the penny from where it started to where it ended up after the composition. For example, suppose the penny starts in square 1 and we do the composition \( H \circ V \). Then the penny ends up in square 4. Now, what single move takes the penny from square 1 to square 4? It is the D move. Is that the only move that will work? Yes. Therefore, starting in square 1, \( H \circ V = D \).

Have the students try:
   a) If a penny is in square 2, what does \( H \circ D \) equal? Answer: V
   b) If a penny is in square 3, what does \( V \circ V \) equal? Answer: I

5. Have students do the Penny Moves Closure Activity worksheet. You may want the students to work in groups and bring the class together to discuss the tasks at various times.

Worksheet Solutions, Discussion, and Extensions

Task 1A
Does the answer to \( H \circ V \) depend on which square the penny starts in? Show how you determined your answer.

No, from any starting square \( H \circ V = D \)

Task 1B
Does the answer to \( D \circ V \) depend on which square the penny starts in? Show how you determined your answer.

No, from any starting square \( D \circ V = H \)

Task 1C
See if any other composition pairs depend upon which square the penny starts in. What can you conclude about the need to know which square the penny starts in?

For any composition pair, the result does not depend on the starting square.

Task 1D
Why do you think your conclusion in Task 1C is true?

This might be challenging for students to explain. The starting square does not matter because the relative movement is the same no matter the starting position. For example, a horizontal move composed with a vertical move will always result in a diagonal movement no matter whether the horizontal movement is left or right or whether the vertical movement is up or down.

Task 2A
To determine if this set of moves is closed under composition, we can create a table to show the results of every possible composition pair by creating a table. One square of the table has been filled in for you: H\(\circ\)V=D. For every square in the table, the move on the left side column is the first move, and the move above (i.e. the top row) is the second move.

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Task 2B
Are the penny moves closed under the operation of composition? Explain how the table you created in Task 2A supports your answer.

Yes, the penny moves are closed under composition. Every combination of moves results in another legal move. This is shown in the table since the table contains every combination of moves and the answers in the table are all legal moves.
Task 2C
What other observations can you make about the penny moves under composition? Hint: you might look for patterns in the table.

Some possible observations:
- One diagonal consists of all I’s while the other diagonal consists of all D’s.
- Each sub-square contains only two of the moves. For example, the upper left quadrant only contains I and H.
- Each row and column contains each of I, H, V, and D exactly once.

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Task 3A
Put the penny in square 1. Then do the following moves and write down the square the penny ends up in. What single move is the same as all the moves you did? $D \circ D \circ V \circ H \circ I \circ D \circ H$

$D \circ D \circ V \circ H \circ I \circ D \circ H = H$

Task 3B
Repeat Task 3A, but start the penny in different squares. Does it matter what square you start in?

No, it does not matter what square the penny starts in.

Task 3C
Is there any sequence of H, D, I, and V moves that does not result in a single move of H, D, I, or V? How did you determine your answer?

No.

Task 3D
How is closure related to your answer in Task 3C?

Since any two moves always result in another legal move and since any string of moves can be broken down into pairs of moves, any string of moves will result in one of the four legal moves.

Extensions
- Why are there I’s on one of the diagonals?

Each square on this diagonal is a move composed with itself. Since moves always “undo” each other, the result is I – the penny remains in its starting position.

- Why are there D’s on one of the diagonals?

Each square on this diagonal represents a composition of an H and a V move, or an I and D move. This always results in a D move.

- How are the penny moves related to addition and multiplication?

Whenever the I move is composed with another move, the result is always the other move. For example, $I \circ M = M$. Thus, I is often called the “identity”. The operations of both addition and subtraction have an identity. Zero is the identity for addition and 1 is the identity for subtraction.
Each penny move has an inverse move. An inverse move is a move that returns the penny to the position from which it started. For example, each move is its own inverse. If an $H$ move is followed by an $H$ move, this is the same as an $I$ move, meaning the penny did not change its initial position. In other words, when a move is composed with its inverse, the result is the identity. For addition, a number’s opposite is its inverse. For example, 2 and -2 are opposites and therefore are inverses under addition. Thus $2 + (-2) = 0$ with zero being the identity for addition. A number’s inverse for multiplication is its reciprocal. For example, $\frac{1}{4} \times 4 = 1$ with 1 being the identity under multiplication.

Another property of addition and multiplication that is modeled in the penny move table is commutativity. With the penny moves, the order in which the moves are composed does not matter. That is, $D \circ V = H = V \circ D$.

It can also be shown that the penny moves exhibit the associative property. Students could work to verify this. Again, the associative property is a property of addition and multiplication, but not of subtraction and division.

The penny moves under composition form what is called a “group” in mathematics. The patterns in the table give some indication of the structure of this group. Groups are an important structure in abstract algebra.