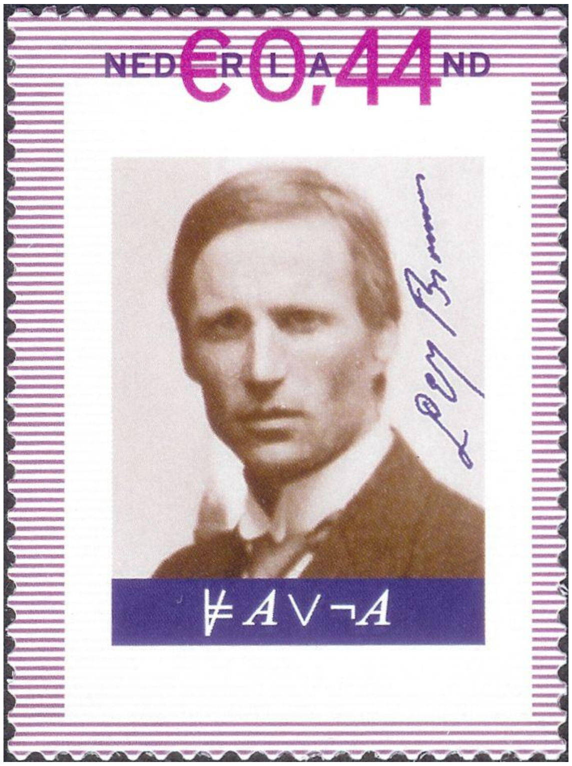

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Luitzen Egbertus Jan Brouwer

(1881 – 1966)

- Dutch mathematician who graduated from the University of Amsterdam.
- Focus in topology, set theory, measure theory, and complex analysis.
- Early proponent for mathematical *intuitionism* as opposed to the traditional mathematical formalism.
- *Intuitionism* is defined by Wikipedia as "… an approach to mathematics as the constructive mental activity of humans. That is, mathematics does not consist of analytic activities wherein deep properties of existence are revealed and applied. Instead, logic and mathematics are the application of internally consistent methods to realize more complex mental constructs.
- Known to his friends as Bertus.
- Killed by a car when crossing the street to his house.

(1912) Brouwer’s Fixed Point Theorem :
Every continuous function $f: B^n \to B^n$ of an *n*-dimensional ball to itself has a fixed point (a point $x \in B^n$ with $f(x) = x$).

Before proving Bertus’s fixed-point theorem, we will need to build up a few definitions and ideas.

Recall from our previous work with Euler’s formula that:

$$\sum_{v \in V} d(v) = 2|E|$$

Where $d(v)$ represents the degree $d(v)$ of a vertex v, V is the set of all vertices in a finite simple graph G, and E is the set of all edges in G.

Proof:
Consider $S \subseteq V \text{x} E$, where S is the set of pairs (v, e) such that $v \in V$ is an end-vertex of $e \in E$. We can count the set S in two different ways. One method is $\sum_{v \in V} d(v)$, since every vertex contributes $d(v)$ to the count. The other method is $2|E|$, since every edges has two vertices. ■

Definition: A *triangulation* is the partitioning of a surface or polygon into a set of triangles, where each edge of a triangle is shared by two adjacent triangles.

Sperner's Lemma: Suppose that some triangle with vertices V_1, V_2, V_3 is triangulated. Assume that the vertices in the triangulation get "colors" from the set {1,2,3} such that V_i receives the color i (for each i), and only the colors i and j are used for vertices along the edge from V_i to V_j (for $i \neq j$), while the interior vertices are colored arbitrarily with 1,2, or 3. Then in the triangulation, there must be a small "tricolored" triangle, which has all three different colors.

Proof:

We can prove a statement stronger than the existence of a tricolored triangle; we can show that the number of tricolored triangles is always odd.

First, construct a triangulation of the triangle $V_1V_2V_3$. Now we will create a partial dual graph to this triangulation by only joining vertices in the dual graph that cross an edge that has end vertices with the different colors 1 and 2. This partial dual graph has degree of 1 at all the vertices that correspond to the tricolored triangles, degree of 2 at all the triangles that only contain the colors 1 and 2, and degree of 0 at all the triangles that do not have both colors 1 and 2. Therefore, the only tricolored triangles correspond to vertices of odd degree.

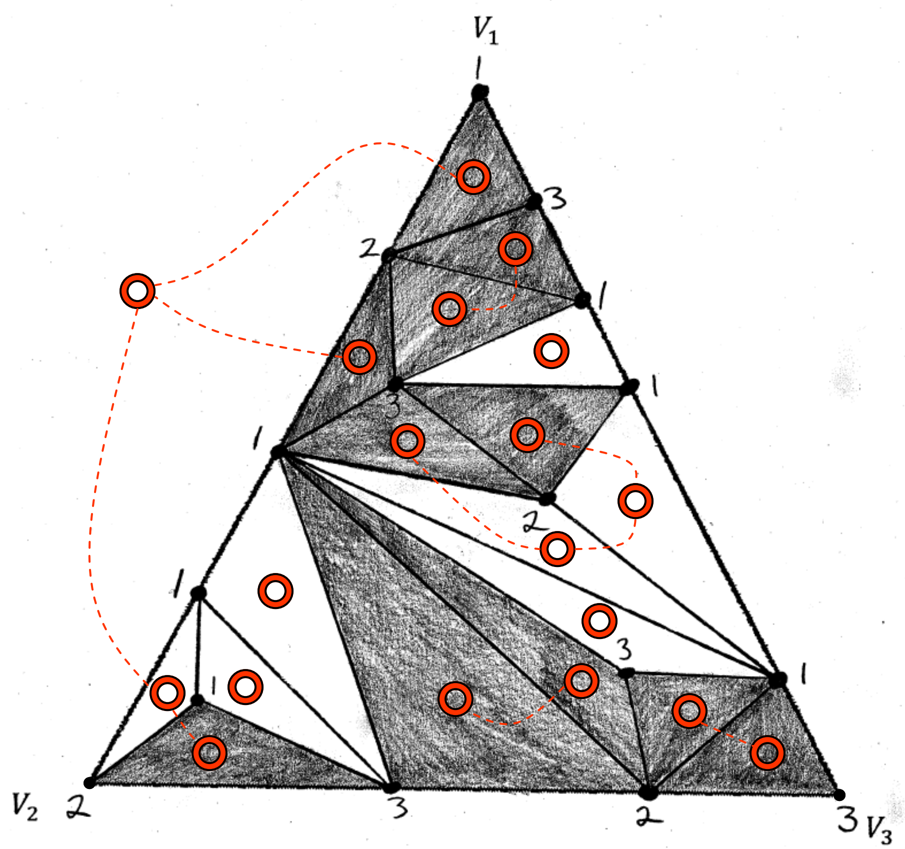
Recall that the dual of a graph contains a vertex that corresponds to the outer region.
Claim: The outer region vertex has an odd degree.
Notice that along our edge from V_1 to V_2 there is an odd number of changes from 1 to 2. Therefore, the partial dual graph has are an odd number of edges crossing the edge from V_1 to V_2. Also, observe that neither of the other two edges from V_1 to V_3 and V_2 to V_3 have both colors 1 and 2. By construction of our partial dual graph, we know that it will never cross the edges from V_1 to V_3 and V_2 to V_3 since we only cross edges of the triangulation that have both colors 1 and 2. Thus, the only edges connected to the vertex corresponding to the outer region cross the edge from V_1 to V_2, which implies that this vertex is of odd degree.

Recall that following:

$$\sum_{v \in V} d(v) = 2|E|$$

From this equation, we can see that the number of odd vertices in any finite graph is even. Thus, the number of odd degree vertices inside the triangle $V_1V_2V_3$ is even since there is only one vertex outside of the triangle $V_1V_2V_3$ with odd degree. Recall that the vertices with odd degree inside the triangle $V_1V_2V_3$ correspond to the tricolored triangles. Hence, the number of tricolored triangles in the triangulation of the triangle $V_1V_2V_3$ is odd. ■

To better understand the proof of *Sperner's Lemma* we can follow the proof through an example triangle $V_1V_2V_3$, and show that the number of tricolored triangles inside triangle $V_1V_2V_3$ is in fact odd.



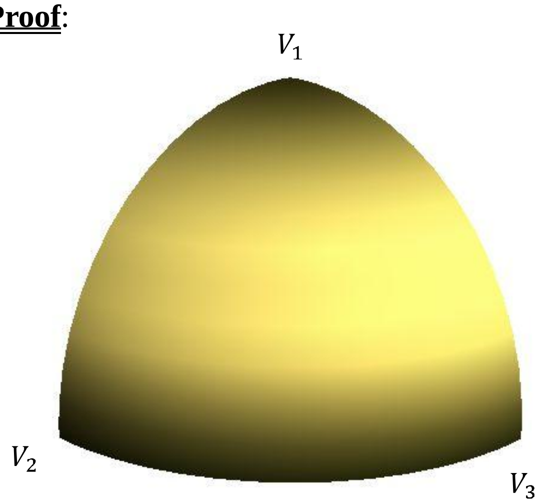
Now observe the degree vertices of the partial dual graph from above.

- The degree of the vertices in tricolored triangles is 1.
- The degree of the vertices in triangles with only vertices 1 and 2 is 2.
- The degree of the vertices in triangles with only vertices 1 and 3 is 0.
- The degree of the vertex representing the outer region is 3 (which is odd).

We will now use *Sperner's Lemma* to help us prove *Brouwer's Fixed Point Theorem.*

Brouwer's Fixed Point Theoremu: Every continuous function $f: B^n \rightarrow B^n$ of an n-dimensional ball to itself has a fixed point (a point $x \in B^n$ with $f(x) = x$).

Proof:



Let $V_1V_2V_3$ be a triangle that covers the surface of a quadrant of the unit sphere, Δ. Then each side of the triangle is equidistant away from its opposite vertex.

Suppose we are given a continuous function $f: \Delta \rightarrow \Delta$ that maps every point in Δ to a point in Δ. We can now define a coloring of the points in Δ in the following manner:

1) For vertices $v \in \Delta$, if the distance from $f(v)$ to V_1 is greater than the distance from V_1 to v (i.e. $|V_1 - f(v)| > |V_1 - v|$) then v is color 1.
2) For vertices $v \in \Delta$ not color 1, if the distance from $f(v)$ to V_2 is greater than the distance from V_2 to v (i.e. $|V_2 - f(v)| > |V_2 - v|$) then v is color 2.
3) For vertices $v \in \Delta$ not color 1 or 2, if the distance from $f(v)$ to V_3 is greater than the distance from V_3 to v (i.e. $|V_3 - f(v)| > |V_3 - v|$) then v is color 3.

Observation
Notice that every point along the side V_2V_3 of Δ is equidistant from the vertex V_1 of Δ. Furthermore, each $v \in V_2V_3$ is the maximum distance any point inside Δ can be from the vertex V_1. Thus the image of any $v \in V_2V_3$, $f(v)$, will always be closer to V_1 than v. Hence there can be no $v \in V_2V_3$ such that v can be color 1.
Following a similar argument, we can see that every $v \in V_1V_3$ can only be colors 1 or 3, and every $v \in V_1V_2$ can only colors 1 or 2.

Now create a triangulated mesh on the triangle Δ. By *Sperner's Lemma* there must exist at least one triangle, Δ_1, in this mesh that has vertices colors 1, 2, and 3 (tricolored triangle).

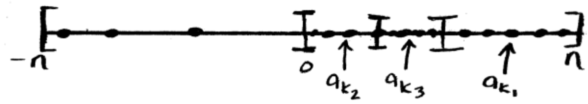
Then construct another triangulated mesh inside Δ_1. Again by *Sperner's Lemma* we know there exists at least one triangle, Δ_2, in this mesh that has vertices are colors 1, 2, and 3 (tricolored triangle). If we repeated this process n times we can find smaller and smaller tricolored triangles $\Delta_1, \Delta_2, \dots, \Delta_n$.

Recall

<u>Bolzano-Weierstrass Theorem</u>: Every bounded sequence contains a convergent subsequence.

<u>Sketch of Proof</u>:

Let (a_k) be a bounded sequence in the interval $[-n, n]$ where $n > 0$. Then $|a_k| < n$ for all $k \in \boldsymbol{N}$. Now bisect the interval into two separate intervals $[-n, 0]$ and $[0, n]$. One of these intervals must contain an infinite number of the points in the sequence (a_k), that contains some point a_{k_1}. Select the interval which this is the case. Now bisect the chosen interval. Once again, one of these two intervals will contain an infinite number of points of the sequence (a_k), denote one of these points as a_{k_2}. Repeating this process will result in a nested sequence of closed intervals. Then by the *Nested Interval Property* we find an $x \in \boldsymbol{R}$ that is in each of these closed intervals. With a little work we can show this x is the limit point of the subsequence (a_{k_i}).



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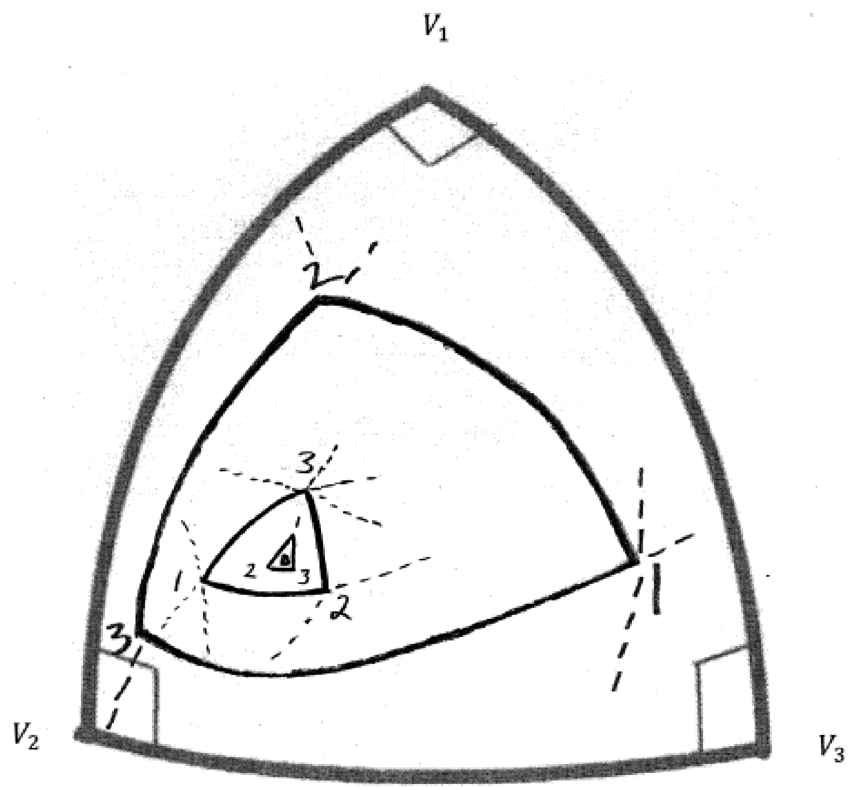
We can use *Bolzano-Weierstrass Theorem* to help us show that this sequence of tricolored triangles is convergent.



By simply repeating *Sperner's Lemma* we can see that we are sectioning our mesh an infinite number of times, which basically mimics the idea of the proof of the *Bolzano-Weierstrass Theorem*. Therefore, a subsequence of the sequence of tricolored triangles converges to a point, v_0 that has all three colors 1, 2, and 3.

Since the subsequence of tricolored triangles converge to v_0 we know that each of the subsequences, $(v_{1_k}), (v_{2_k}), (v_{3_k})$, of this convergent sequence must also converge to v_0.

Now we want to show this point v_0 is a fixed point in Δ (i.e. $f(v_0) = v_0$).
Assume contrary. Suppose v_0 is not a fixed point, then $f(v_0) \neq v_0$. Since $f(v_0) \neq v_0$ we know $f(v_0)$ is closer to one of the vertices, V_1, V_2, or V_3, than v_0.
WLOG assume that $f(v_0)$ is closer to the vertex V_1 than v_0 is. Thus, we know $|V_1 - v_0| > |V_1 - f(v_0)|$. Using this inequality, let $0 < \varepsilon < |V_1 - v_0| - |V_1 - f(v_0)|$. Now by the continuity of f, there exists some $\delta > 0$ such that for any vertex v within a δ-neighborhood of v_0, for any $\varepsilon > 0$ $|f(v) - f(v_0)| < \varepsilon$.

Now we know there exists vertices of the form v_{1_j} within a δ-neighborhood of v_0 since the subsequence (v_{1_k}) converges to v_0. Therefore, choose $0 < \delta < |V_1 - f(v_0)|$ then $\left|v_{1_j} - v_0\right| < \delta$ implies for any $\varepsilon > 0$ that $\left|f\left(v_{1_j}\right) - f(v_0)\right| < \varepsilon$.

Using the continuity of f and the triangle inequality, we can create a contradiction.

$$
\begin{aligned}
0 &< |V_1 - v_0| - |V_1 - f(v_0)| \\
&\leq \left|V_1 - v_{1_j}\right| + \left|v_{1_j} - v_0\right| - \left(\left|V_1 - f(v_{1_j})\right| + |V_1 - f(v_0)|\right) \\
&= \left|V_1 - v_{1_j}\right| + \left|v_{1_j} - v_0\right| - \left|V_1 - f(v_{1_j})\right| - |V_1 - f(v_0)| \\
&< \left|V_1 - v_{1_j}\right| + \delta - \left|V_1 - f(v_{1_j})\right| - |V_1 - f(v_0)| \\
&< \left|V_1 - v_{1_j}\right| - \left|V_1 - f(v_{1_j})\right|
\end{aligned}
$$

This implies that $\left|V_1 - v_{1_j}\right| - \left|V_1 - f\left(v_{1_j}\right)\right| > 0$. This means $\left|V_1 - v_{1_j}\right| > \left|V_1 - f\left(v_{1_j}\right)\right|$ which tells us $f\left(v_{1_j}\right)$ is closer to V_1 than the vertex v_{1_j}, but this contradicts our assumption that v_{1_j} has color 1. Therefore, $f(v_0) = v_0$ and thus Δ has the fixed point v_0.

■

To see visually consider the figure below: