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Complex Analysis
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WEIERSTRASS M-TEST

Before proving the *Weierstrass M-Test* we will need to gather a few tools.

THEOREM **(CAUCHY CRITERION FOR CONVERGENCE):**

Suppose $\{Z_n\}$ is a sequence of complex numbers for $n \in \mathbb{N}$. Then $\{Z_n\}$ converges if and only if for any $\varepsilon > 0$ there exists an $N \in \mathbb{Z}$ such that $|Z_n - Z_m| < \varepsilon$ for every $m, n \in \mathbb{Z}$ such that $m > N, n > N$.

Any sequence that satisfies the *Cauchy Criterion* is known as a Cauchy sequence. The above theorem also shows that every convergent sequence is Cauchy, and every Cauchy sequence is convergent.

COROLLARY 1:

If $\{Z_n\}$ is a Cauchy sequence that converges to Z , and N is chosen such that $|Z_n - Z_m| < \varepsilon$ for every $m, n \in \mathbb{Z}$ such that $m > N, n > N$, then for each $n > N$, $|Z_n - Z| \leq \varepsilon$.

PROOF:

This proof is rather straightforward. Let $m \rightarrow \infty$ in the inequality $|Z_n - Z_m| < \varepsilon$. It follows from this that $|Z_n - Z| \leq \varepsilon$. ■

COROLLARY 2:

The series $\sum_{k=0}^{\infty} a_k$ converges if and only if for any $\varepsilon > 0$ there exists an N such that $|\sum_{k=n+1}^m a_k| < \varepsilon$ for every $m, n \in \mathbb{Z}$ such that $m > n > N$.

Now it is possible to prove the generalized *Comparison Test*, also known as the *Weierstrass M-Test*.

THEOREM**(WEIERSTRASS M-TEST FOR CONVERGENCE OF FUNCTIONAL SERIES):**

Suppose $\{f_k\}$ is a sequence of real- or complex-valued functions on some set E . Also, suppose that $\sum_{k=0}^{\infty} M_k$ is a convergent series where M_k are real non-negative terms. If $|f_k(z)| \leq M_k$ for all k greater than some number N and for all z in some set E , then it follows that the series $\sum_{k=0}^{\infty} f_k(z)$ converges uniformly on E .

PROOF:

Since $\sum_{k=0}^{\infty} M_k$ is Cauchy, we can choose a number $M > N$ such that for any m and n that satisfy $m > n > M$ we get that $\sum_{k=n+1}^m M_k < \varepsilon$. Then we see that for z in the set E that our series $\sum_{k=0}^{\infty} f_k(z)$ is also Cauchy, since

$$\left| \sum_{k=n+1}^m f_k(z) \right| \leq \sum_{k=n+1}^m |f_k(z)| \leq \sum_{k=n+1}^m M_k < \varepsilon$$

Therefore, $\sum_{k=0}^{\infty} f_k(z)$ converges for every $z \in E$. Let us say that $\sum_{k=0}^{\infty} f_k(z)$ converges to the function $F(z)$.

Now, we want to show that $\sum_{k=0}^{\infty} f_k(z)$ converges uniformly to $F(z)$. Observe that we can rewrite

$$\left| \sum_{k=n+1}^m f_k(z) \right| \leq \sum_{k=n+1}^m |f_k(z)| \leq \sum_{k=n+1}^m M_k < \varepsilon$$

in terms of partial sums

$$\left| \sum_{k=0}^m f_k(z) - \sum_{k=0}^n f_k(z) \right| < \varepsilon$$

for all $z \in E$, and where $m > n > N$. Then applying the 1st Corollary of the **Cauchy Criterion Theorem**, we see that

$$\left| F(z) - \sum_{k=0}^n f_k(z) \right| \leq \varepsilon$$

for $z \in E$, and where $m > n > N$. Thus, the uniform convergence is shown. ■

So now, if we look at the *Comparison Test*, we see that it is a special case of the *Weierstrass M-Test* where $f_k(z)$ is just a constant function.

THEOREM

(COMPARISON TEST):

Suppose we have the terms a_k such that $|a_k| \leq M_k$ for all $k \in \mathbb{Z}$, $k > N$ for some number N . Then if the series $\sum_{k=0}^{\infty} M_k$ converges, the series $\sum_{k=0}^{\infty} a_k$ converges as well.

Since we know some of the ideas behind the *Weierstrass M-Test*, we can now begin to look at some of its applications. We will first consider an application of the *Weierstrass M-Test* in the set of \mathbb{R} , before moving into applications within the set of \mathbb{C} .

Example

Show that the real-valued series

$$\sum_{k=1}^{\infty} \frac{1}{4^k} \sin\left(\frac{x}{3^k}\right)$$

is uniformly convergent.

The *Weierstrass M-Test* gives us the ability to show this without considering any limits. First, we observe that for any $x \in \mathbb{R}$, $|\sin\left(\frac{x}{3^k}\right)| \leq 1$ for all k . Then it is easy to see that $\left|\frac{1}{4^k} \sin\left(\frac{x}{3^k}\right)\right| \leq \frac{1}{4^k}$. So now let $M_k = \frac{1}{4^k}$.

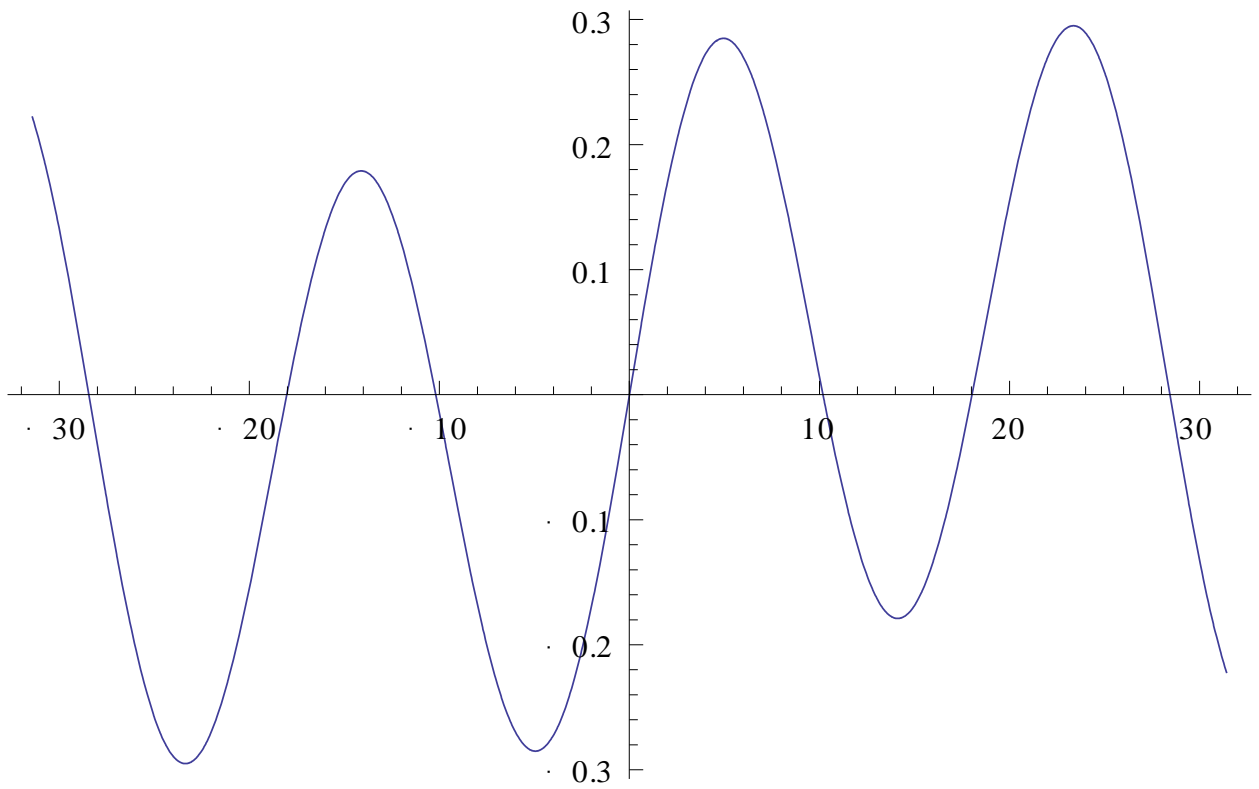
Now we want to show that the series $\sum_{k=1}^{\infty} \frac{1}{4^k}$ (our series $\sum_{k=1}^{\infty} M_k$) is convergent. This series converges to $\frac{1}{3}$ by the following **Lemma**.

LEMMA:

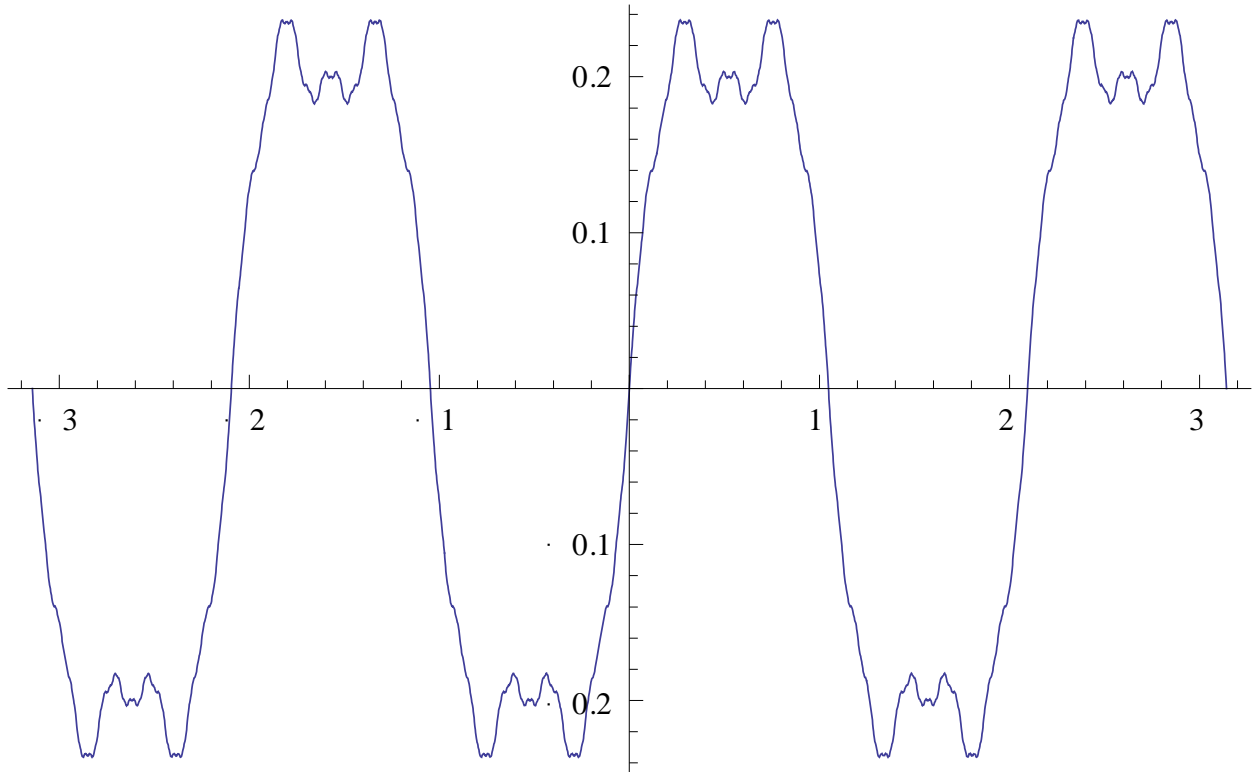
The series $\sum_{k=0}^{\infty} a^k$ converges to $\frac{1}{1-a}$ if $|a| < 1$.

So we now have $\sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{4}{3} = 1 + \sum_{k=1}^{\infty} \frac{1}{4^k} = 1 + \frac{1}{3}$. Hence, $\sum_{k=1}^{\infty} \frac{1}{4^k} = \frac{1}{3}$ (our series $\sum_{k=1}^{\infty} M_k = \frac{1}{3}$). Now by the *Weierstrass M-Test* we see the series $\sum_{k=1}^{\infty} \frac{1}{4^k} \sin\left(\frac{x}{3^k}\right)$ is uniformly convergent on \mathbb{R} . ■

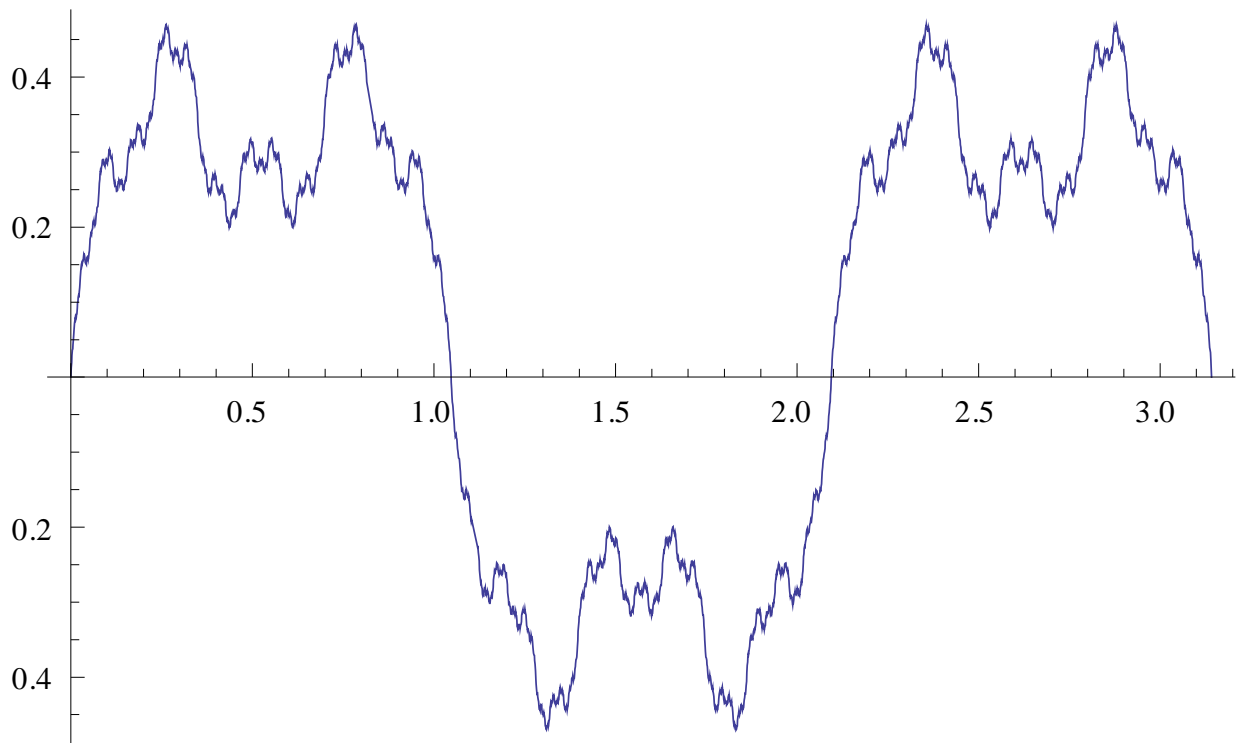
Below we see the behavior of the series $\sum_{k=1}^{\infty} \frac{1}{4^k} \sin\left(\frac{x}{3^k}\right)$, and we can see how $\frac{1}{4^k}$ dampens the amplitude of the periodic sin function.



We can also look at the series $\sum_{k=1}^{\infty} \frac{1}{4^k} \sin(3^k x)$ graphically and see how this change effects the dampening effect of $\frac{1}{4^k}$.



We can also look to see how altering the dampening coefficient may affect the graph. Let the dampening coefficient be $\left(\frac{1}{2.5}\right)^k$



Now to consider an application of the *Weierstrass M-Test* in the set of \mathbb{C} .

Example

Show that the exponential function $f(z) = e^z$ is uniformly convergent on any bounded set $S \subset \mathbb{C}$.

Recall that e^z can be rewritten as the series $\sum_{k=0}^{\infty} \frac{z^k}{k!}$. Now we will show that this series is uniformly on some disk D of radius r centered at the origin. To show this we must find some M_k such that $\left| \frac{z^k}{k!} \right| \leq M_k$ for all $z \in D$. Recall that $|z^k| \leq |z|^k$, and that $|z| \in \mathbb{R}$. So let $|z| \leq r \in \mathbb{R}$. Then it follows that $\left| \frac{z^k}{k!} \right| \leq \frac{|z|^k}{k!} \leq \frac{r^k}{k!}$. We see that $\frac{r^k}{k!} \in \mathbb{R}$, so now let $M_k = \frac{r^k}{k!}$.

We may be able to apply the *Weierstrass M-Test* if we can show that the series $\sum_{k=0}^{\infty} M_k$ converges. If we use the **Ratio Test** we can prove that $\sum_{k=0}^{\infty} M_k$ is convergent. So now recall:

RATIO TEST:

Given a series $\sum_{k=0}^{\infty} a_k$, find

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = L$$

- If $L > 1$, the series diverges
- If $L < 1$, the series converges
- If $L = 1$ or the limit fails to exist, then the test is inconclusive.

So now we see that $\lim_{k \rightarrow \infty} \frac{M_{k+1}}{M_k} = \lim_{k \rightarrow \infty} \frac{\frac{r^{k+1}}{(k+1)!}}{\frac{r^k}{k!}} = \lim_{k \rightarrow \infty} \frac{r}{k+1} = 0$. Thus by the **Ratio Test** we see that

the series $\sum_{k=0}^{\infty} M_k$ converges. Then by the *Weierstrass M-Test* we see that $\sum_{k=0}^{\infty} \frac{z^k}{k!}$ is uniformly convergent on some disk D of radius r centered at the origin. ■

References

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