A partial chainability condition for regions obtained from simple dense canals

Steven C. Leth

June 2010

We will call an internal subset of the nonstandard plane **IL-chainable** (for "infinitesimally linearly chainable") if it can be covered by a hyperfinite collection of open sets $G_0, ..., G_n$ of infinitesimal diameter with the "linear chain" condition that $G_i \cap G_j$ is nonempty iff $|i-j| \leq 1$.
If $\delta > 0$ we will call a standard subset of the plane, or an internal subset of the nonstandard plane δ-**chainable** if it can be covered by a finite (or hyperfinite) collection of open sets $G_0, ..., G_n$ of diameter less than δ with the "linear chain" condition that $G_i \cap G_j$ is nonempty iff $|i-j| \leq 1$.

If A is a standard set and ${}^*A$ is IL-chainable, then A is chainable in the classic sense. To see this we note that if ${}^*A$ is IL-chainable then for any given standard $\delta > 0$ we can transfer to the standard universe the statement that A can be covered by a chain with mesh less than δ. Conversely, if A is chainable in the classic sense then we can transfer to the nonstandard universe the statement that for all $\delta > 0$ there is a chain covering A with mesh less than δ, and so this is true of ${}^*A$ using an infinitesimal for δ.

But we are mostly interested here in various internal sets being IL-chainable. This corresponds to a result about the existence of corresponding standard sets that are δ-chainable, where the set depends on δ.

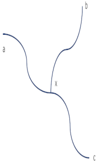
By a **simple dense canal** in E we mean an infinite arc C contained in the complement of E such that C is the image of a one-to-one continuous function c from $[0, \infty)$ to $\mathbb{R}^2$ that has the property that for all $\varepsilon > 0$ there exists t_0 such that for all $t \in [t_0, \infty)$, $c(t)$ is on a "transverse cross cut" of distance less than ε (i.e. a segment whose endpoints are in E on opposite sides of the arc and whose length is less than ε), and that is dense in E (i.e. is such that $\overline{C} - C = E$). I am paraphrasing Mayer's wording of

Sieklucki's definition. (Question: It seems to sometimes be used to mean the open set itself and sometimes a defining arc such as this?)

Simple dense canals seem to be fertile ground for the use of nonstandard methods.

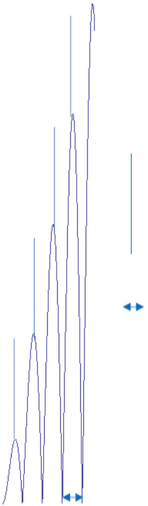
Definition

We will say that a set $A \subset {}^*\mathbb{R}^2$ contains a **Y-set** if there exist four points a, b, c, and x in A, arcs C_{ax}, C_{bx}, and C_{cx} in A intersecting only at x, from a to x, b to x, and c to x, respectively, such that none of the points a,b, or c are infinitesimally close to any point on the arcs joining the others to x (thus, for example, no point of C_{ax} is infinitesimally close to b or to c). If $\delta > 0$ we will say that a set A in the plane contains a **size** δ **Y-set** if there exist four points a, b, c, and x in A, and arcs C_{ax}, C_{bx}, and C_{cx} in A intersecting only at x from a to x, b to x, and c to x, respectively, such that none of the points a,b, or c are within δ of any point on the arcs joining the others to x (thus, for example, no point of C_{ax} is within distance δ of b or c).



Question: Is there a better name? Is there already a name for this? Is something like "δ-thick triod" better?

If a set contains a size δ Y-set then it is easy to see that it cannot be δ-chainable. The converse is not true. For each n there exist sets that are not $n\delta$ chainable but contain no size δ Y set. The next slide shows a picture example. Similarly if a set is IL-chainable then it contains no Y-set. However it is not known whether the converse is true or not. (It seems very unlikely, but I do not yet have a counterexample). Extending the picture counterexample in the simplest possible way as n increases does not work, as it yields a graph that has height greater than any standard natural number within a bounded x domain, and we are interested here in examples in which the standard part of all points exists, i.e. in sets contained inside some standard disk.



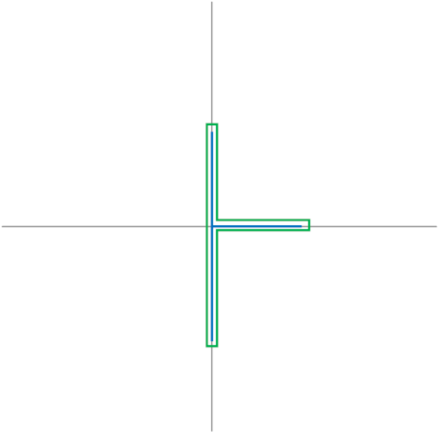
If δ is smaller than this distance then the set is not δ chainable

This set contains no Y-set of size greater than this distance.

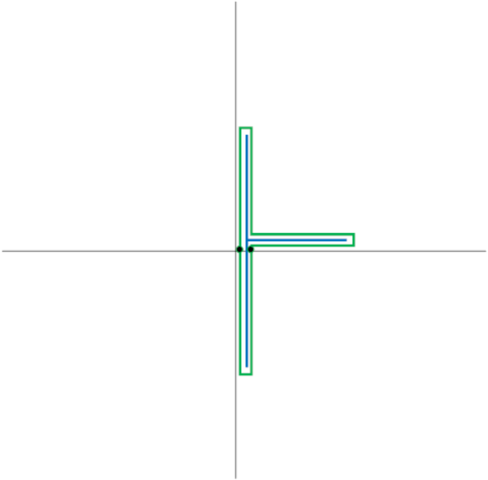
Theorem

Let $V \subset {}^*\mathbb{R}^2$ be a bounded (internal) region (i.e. contained in some ${}^*B(0, r)$ for a standard real $r > 0$) bounded by a simple closed curve S and suppose that $\overline{V}$ contains no standard points, $st(V)$ does not disconnect the plane, and that there exists an arc A of infinitesimal length on S such that ${}^*st(V) \cap S \subset A$. Then V contains no Y-set.

If we remove the condition that V contains no standard points there are simple counterexamples. For example, we may let the blue line segments below form a standard set and the green border be S, where every point of S stays within an infinitesimal of the blue set. The blue set itself is clearly a Y-set, and the other conditions of the theorem are satisfied.



If we move the blue set over and up by an infinitesimal amount, we cannot surround it by a border that intersects ${}^*st(V)$ only on one small arc.



Sketch of proof: Suppose that such a set V contains a Y-set. We may choose our a, b, c, and x such that the small arc A is not near a, b, or c. We find balls B_a, B_b, B_c satisfying the following conditions:

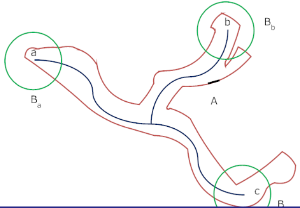
i) $a \in {}^*B_a$, $b \in {}^*B_b$, and $c \in {}^*B_c$.

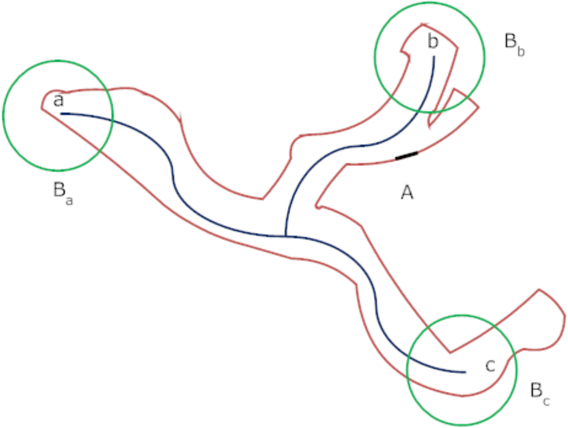
ii) Each of these three balls have non-infinitesimal radius.

iii) The arc A does not intersect $\overline{B_a}, \overline{B_b}$, or $\overline{B_c}$

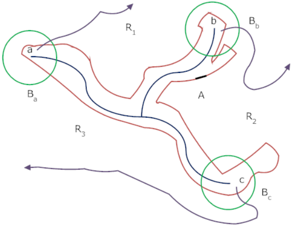
iv) every point on $\overline{B_a}$ is not near any point on C_{bx} or C_{cx}, every point on $\overline{B_b}$ is not near any point on C_{ax} or C_{cx}, and every point on $\overline{B_c}$ is not near any point on C_{ax} or C_{bx}

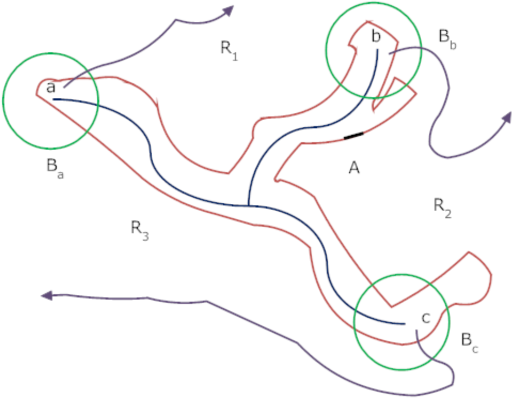
v) The interior of each ball contains points not disconnected from infinity by $st(Y)$ together with the closure of the other two balls. For example, the interior of B_a contains points not disconnected from infinity by $st(Y) \cup \overline{B_b} \cup \overline{B_c}$.




a
b
c
B_a
B_b
B_c
A

We let Y' be the portion of the Y-set obtained by starting at x and following each of the three paths to the first points of intersection with the three balls. We then define internal infinite arcs Γ_a, Γ_b and Γ_c that, along with Y' divides the nonstandard plane into three regions as shown. We consider connected components of $^*\mathbb{R}^2 - (S\cup \Gamma_a\cup \Gamma_b\cup \Gamma_c))$. One of these is V, which intersects all three of our newly constructed regions, but this is the only connected component of this set that can intersect more than one of these regions. The set $^*st(Y')$ is connected, but there must be standard points of it in two different regions. We show that one of these standard points must be contained in a set whose boundary is completely contained in the complement of $^*st(Y')$, contradicting the fact that it is connected.



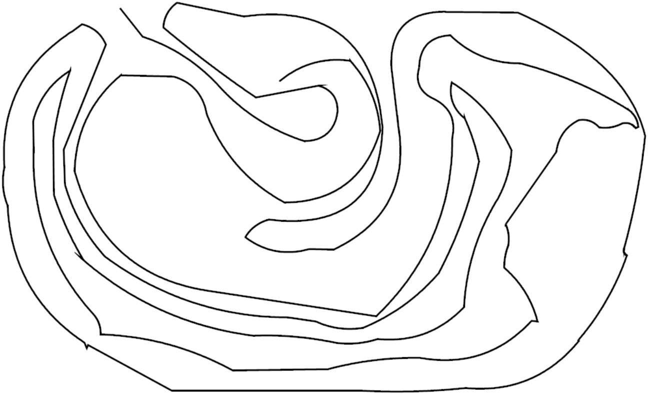

R$_1$
b
B$_b$
a
B$_a$
A
R$_2$
R$_3$
c
B$_c$

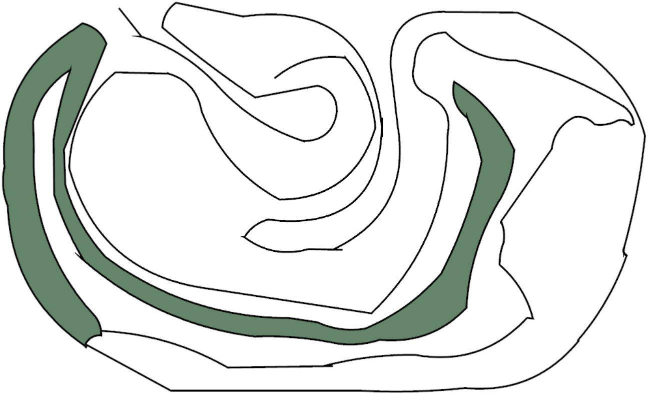
We want to use this result to show that we can "close off" portions of a simple dense canal with uncomplicated arcs to obtain sets that contain no Y-set. In the theorem below we use arcs of circles to close off the regions.

Theorem

Let E be a non-separating plane continuum that contains a simple dense canal C. Then there exist points $p_1 \approx p_2$ on ${}^*C$ and an arc A of a circle such that the portion of ${}^*C[p_1,p_2]$ together with $A[p_2, p_1]$ forms a simple closed curve S that is infinitesimally close to every point in E and is such that V_S contains no standard points in E.

We want to close off a portion with an arc of a circle in such a way that the region contains no standard points (and thus, by the previous theorem, contains no Y-set).

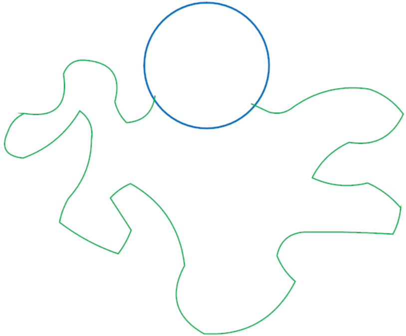




By saturation, we need to show the following standard result:

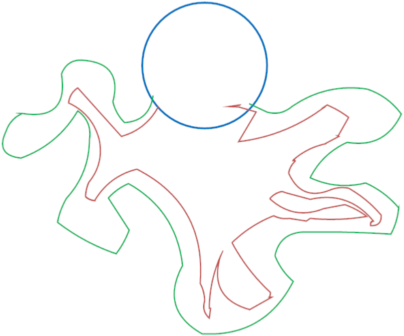
For any finite collection of standard points $\{x_1, x_2, ...x_n\}$ in E and any standard $\delta > 0$ there exist points a and b on C and an arc of some circle A such that $C[a, b]$ is within δ of every point of E, a and b are within δ of each other, and the portion of C from a to b together with $A[b, a]$ forms a simple closed curve whose interior does not contain any of the points $\{x_1, x_2, ...x_n\}$.

Nonstandard methods still seem to be very helpful in proving this standard result. The picture below illustrates why. Suppose that the ball shown is standard, but that the part of ${}^*C$ shown is within δ of every point of E.



Then this must repeat infinitely often with later portions of the curve. The lemma in the next slide makes this more precise.

This allows us to create (many annoying details later) infinitely many disjoint regions that all are within δ of every point of E.



Lemma

Let E and C be as in the theorem, let $\delta > 0$ be a standard real number, and suppose that there exist points u and v in ${}^*C - C$ such that the following conditions are satisfied;
i) both u and v are on the boundary of a standard disk D of radius less than δ.
ii) ${}^*C[u, v]$ is within δ of every point in E.
iii) ${}^*C[u, v] \cap \overline{D} = \{u, v\}$.
We note that condition iii) implies that ${}^*C[u, v]$ together with $L[v, u]$ forms a simple closed curve, which we will call S.
Then one of the following must be true:
a) arbitrarily far out on ${}^*C$ and arbitrarily close to $st(u)$ and $st(v)$, respectively, there exist points u' and v'such that conditions i) through iii) are satisfied if we replace u and v with u' and v' in each statement, and the simple closed curve S' (formed by ${}^*C[u', v']$ together with $L[v', u']$) is contained in V_S (the "inside" region defined by S) or
b) arbitrarily far out on ${}^*C$ and arbitrarily close to $st(u)$ and $st(v)$ there exist points u' and v' satisfying all the same conditions except that S is contained in $V_{S'}$.

Proof.

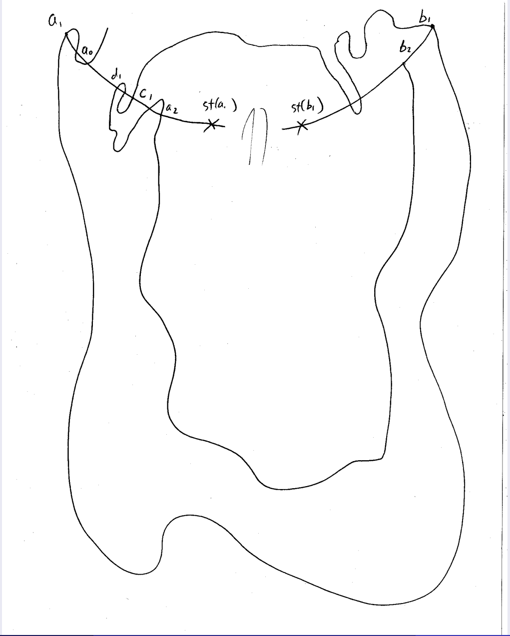
The standard set $F = st({}^*C[u,v])$ is a closed, connected subset of E. We note that no element of F (and hence no element of ${}^*F$ by transfer) intersects the interior of D and all points on S other than those in the interior of D are in the complement of E. Since ${}^*F$ is connected and does not intersect S we see that either ${}^*F \subset V_S$ or ${}^*F \cap V_S = \varnothing$. These two cases will determine which of a) or b) is true in the lemma. We assume the first case only; the proof in the second case is essentially the same. $\square$

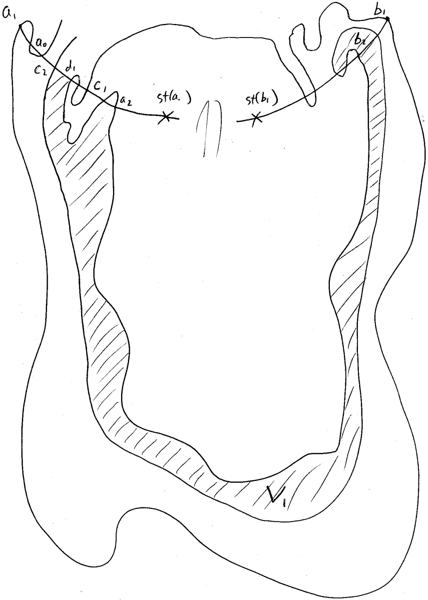
(Continued)

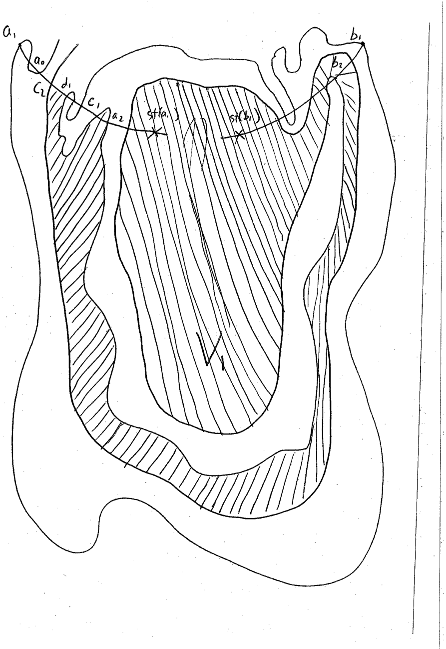
Let x be some standard point in ${}^*F$. So, $x \in V_S$ and is infinitesimally close to but not contained in ${}^*C[u, v]$ (since x is in E). We let η be a positive infinitesimal less than the minimum distance between x and ${}^*C[u, v]$. Then in the nonstandard universe ${}^*C[u, v]$ is witness to the fact that for any standard $\epsilon > 0$ and beyond any standard point in C there exist points u and v on the curve such that properties i) through iii) hold, and such that x is in V_S, x is within ϵ of ${}^*C[u, v]$ and u and v are within ϵ of $st(u)$ and $st(v)$ respectively. Thus, this is true in the standard universe as well by transfer, and so the standard universe satisfies the statement that there are points arbitrarily far out on C such that i)-iii) hold and such that the relevant portion of the curve passes arbitrarily close to x. Then, again by transfer, this is true in the nonstandard universe, and so beyond any point on the curve ${}^*C$ there exist points u' and v' satisfying conditions i) through iii) and such that x is in $V_{S'}$ and such that the minimum distance from ${}^*C[u', v']$ to x is less than η, and u' and v' are within an infinitesimal distance of $st(u)$ and $st(v)$ respectively. These conditions now guarantee that S' is contained in V_S, and condition a) is satisfied.

The goal now, in using the lemma to prove the theorem, is to use the fact that there must exist these repeated portions of the curve arbitrarily far out and arbitrarily close to the standard part of the intersection of the curve with the circle. The diagrams in the next three frames are to accompany the proof of this part from the paper.

Proof.




a_1
a_0
c_2
d_1
c_1
a_2
$st(a_1)$
$st(b_1)$
b_1
b_2
V_1


a₁
a₀
c₂
d₁
c₁
a₂
st(a₁)
st(b₁)
b₁
b₂
V₁

The theorem below is a standard version of this last result.

Theorem

Let E be a non-separating plane continuum that contains a simple dense canal, and let C be an infinite arc (i.e. the image of a one-to-one continuous function from $[0, \infty)$ to $\mathbb{R}^2$) in the complement of E with the property that $\overline{C} - C = E$ and for every $\varepsilon > 0$ there exists a point p on C

such that all points on the arc beyond p are on a transverse cross cut of distance less than ε. Then for any $\delta > 0$ there exist points p_1 and p_2 on C, and an arc of a circle A, such that $C[p_1, p_2]$ together with $A[p_1, p_2]$ forms a simple closed curve S that is within δ of every point in E and is such that V_S contains no size δ Y-set.

The conjecture is that all of the statements involving no Y-sets (or no size δ Y sets) can be improved to IL-chainability (δ chainability). Thus, the immediate next goal is to prove this conjecture:

Let $V \subset {}^*\mathbb{R}^2$ be a bounded region bounded by a simple closed curve (internally) and suppose that $\overline{V}$ contains no standard points, $st(V)$ does not disconnect the plane, and that there exists an arc A of infinitesimal length on $\partial(V)$ such that ${}^*st(V) \cap \partial(V) \subset A$. Then V is IL-chainable.

It seems quite likely that the theorem can be improved to the statement above even if being IL-chainable does not follow from the condition "contains no Y-set." This is because the condition that V contain no standard points and has small intersection with ${}^*st(V)$ is quite strong and precludes, for example, the pursuit of any counterexample in which there are long patches of many thin long loops that would create a standard part with an interior. For example, if we have a part like in an earlier example, but the distances are infinitesimal while the x-direction is noninfinitesimal there will be large intersection of the boundary of V and ${}^*st(V)$.