SOME NOTES ON NONSTANDARD METHODS IN THE PLANE

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ABSTRACT. These notes are intended to provide a starting point for the use of nonstandard methods in continuum theory.

1. Nonstandard Analysis

Nonstandard analysis, first developed by Abraham Robinson in the early 1960's [6], makes use of the fact that there are other *models* besides the usual ones that satisfy all the same mathematical statements that can be made in $First\ Order\ Logic$. While it is possible to develop nonstandard methods purely using ultrafilters without any reference to logic, I believe that the fundamental benefit is lost with this approach. Nonstandard methods have the ability to greatly simplify the discourse and greatly enhance the intuition of standard notions with just a basic understanding of which properties are the same in the nonstandard world and which are different. The most famous "different" property of nonstandard sets - the existence of infinitesimals in the nonstandard reals - is but one of many examples in which there are actual objects that correspond only to limits in the standard setting. The goal is to exploit what is different - the existence of a wide array of "actualized" limits - and also what is the same - all mathematical properties that can be expressed formally in a very precise sense.

There are nonstandard versions of the reals, the plane, the integers and, in fact, every infinite set. We indicate the nonstandard version of a standard set with a "*", for example by "${}^*\mathbb{R}^2$." Intuitively, the nonstandard reals, for example "think" they are the reals in the sense identified above - they satisfy all the same mathematical statements within the framework of first order logic. For example, they satisfy the statement that says that between any two reals there is another real number, expressed by:

$$\forall x \forall y((x < y) \Rightarrow \exists z((x < z) \wedge (z < y))).$$

They even satisfy the statement

$$\forall x((x \in P(\mathbb{R}) \wedge (x \neq \varnothing) \wedge (\exists y(\forall z \in x(z \leq y)) \Rightarrow$$
$$\exists u(\forall z \in x(z \leq u)) \wedge \forall v(\forall z \in x(z \leq v)) \Rightarrow (u \leq v)).$$

This statement says that every nonempty subset of the real numbers that has an upper bound has a least upper bound. Since it is expressible in this way, the

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nonstandard model must also satisfy this statement. And yet, the nonstandard model contains *infinitesimals* -numbers that are not 0 but whose distance to zero is less than every standard positive number. The set of all infinitesimals has an upper bound (the number .4, for example) but has no least upper bound, which seems to contradict what we just said. However, the set of all infinitesimals is not a set that the nonstandard model "recognizes" as a set.

The *internal* sets are the ones that the nonstandard model recognizes as sets, and it is these sets which must have the same properties as the standard model. When we move to the nonstandard world, the beginning of the statement above corresponds to insisting that $\forall x((x \in {}^*P(\mathbb{R})$........, and those elements of ${}^*P(\mathbb{R})$ are exactly the internal sets. Basically these sets are definable inside the nonstandard "universe." Sets such as the infinitesimals are in $P({}^*\mathbb{R})$ but are not in ${}^*P(\mathbb{R})$.

Is it possible to get a more concrete view of what these nonstandard sets look like? When we define the rationals starting with the reals we might use equivalence classes of Cauchy Sequences of rationals, and we identify two sequences $< a_n >$ and $< b_n >$ as being equivalent iff $\lim_{n\to\infty}(a_n - b_n) = 0$. One way to think of a nonstandard universe is that all the elements, including the sets, are sequences like this, but we have a more refined equivalence relation. We use the concept of a *nonprincipal ultrafilter* on the natural numbers to make this definition. An ultrafilter is a collection of subsets (of $\mathbb{N}$ in this case) that is closed under finite intersections and has the property that if A is in the ultrafilter and $A \subseteq B$ then B is in the ultrafilter, and that every set or its complement is in the ultrafilter. A simple example of an ultrafilter is the collection of all sets that contain the number 3. This example is too simple (it is a *principal* ultrafilter). We will use an ultrafilter that contains no finite sets (a *nonprincipal* ultrafilter). We can use the axiom of choice to show that such ultrafilters exist. We could also think of this as a finitely additive $\{0, 1\}$ measure (the sets in the ultrafilter have measure 1, those that are not in the ultrafilter have measure 0).

One way to make ${}^*\mathbb{R}$ now is to let the set consist of all infinite sequences of reals, and define two sequences $< a_n >$ and $< b_n >$ to be equivalent iff $\{n : a_n = b_n\}$ is in the ultrafilter (equivalently, has measure 1 in our $\{0, 1\}$ measure). Similarly $< a_n > \ \leq \ < b_n >$, for example, if $\{n : a_n \leq b_n\}$ is in the ultrafilter. The real number a is now identified with $< a, a, a, a, ... >$, and we can see that the number $< 1, \frac{1}{2}, \frac{1}{3}, ... >$ is an example of an infinitesimal. This same construction applies to sets as well as elements. The internal sets are precisely the ones that are of the form $< A_n >$ where each A_n is a standard set. Two nonstandard sets are equal if the set of indices on which they agree is in the ultrafilter. We generally will take an ultrafilter over a larger (uncountable) index set.

We now try to get the best of both worlds. We work in a language that is quite strong, but not strong enough to characterize sets up to isomorphism, so there are these nonstandard models. When convenient, we use the fact that infinitesimal or infinite elements exist, and yet we often use the *transfer* principle that says every definable statement true in the nonstandard world is also true in the standard world. We usually work inside a part of set theory that contains all the standard objects we are interested in. In this case that would include the plane, all subsets of the plane, and perhaps even all functions from the plane to the plane, or all possible topologies on the plane, etc. We use ultrafilters over larger index sets because this leads to structures that have more *saturation*. Given a cardinality κ a structure is $\kappa-$*saturated* iff every collection of internal sets of cardinality less than κ with the finite intersection property has a nonempty intersection. For example we can see that any structure that is κ saturated for some uncountable κ (so that

any countable collection of internal sets with the finite intersection property has a nonempty intersection) must contain infinitesimals. We consider the collection of internal sets $\{(0,1),(0,\frac{1}{2}),(0,\frac{1}{3}),....\}$. This set clearly has the finite intersection property, and any element in the intersection of all these sets is an infinitesimal.

We may apply standard functions to nonstandard objects in the obvious way: $f(< a_\xi >) =< f(a_\xi) >$ (here we assume that ξ ranges over some appropriate index set). Although there is a notion of internal functions that is analogous to internal sets, here we will usually be concerned only with applying standard functions to our nonstandard objects. While the nonstandard version of a standard set A is denoted ${}^*A$ we will not follow the same convention with functions, choosing instead to write $f(p)$ to denote the standard function if p is standard and the nonstandard version of the standard function if p is nonstandard.

2. Some basic tools for working in the nonstandard plane

We write $st(a)$ for the unique real number (if a is a nonstandard real) or the unique point in the plane (if a is in the nonstandard plane), that is within an infinitesimal distance of a, assuming that any such number or point exists (there are nonstandard real numbers larger than every standard number, and these have no standard part). If A is a subset of the nonstandard reals or the nonstandard plane then $st(A)$ is the set of all standard parts of A.

We write $a \approx b$ if $\|a-b\|$ is infinitesimal.

We give some simple nonstandard equivalents of standard set properties. Here we are only considering sets in the plane, but these definitions are true in all metric spaces and in most cases have simple analogues in Hausdorff spaces. Most of these results go back to Robinson. Any proofs not included can be found in a number of sources including [2],[4], or [5].

Proposition 1. *i) The set A is open iff for all $a \in A$, whenever $b \approx a$ then $b \in {}^*A$.*

ii) The set A is closed iff whenever $a \in {}^*A$ and $st(a)$ exists then $st(a) \in A$.

iii) The set A is compact iff every point of ${}^*A$ is near some standard point of A.

Proof. i) (An example of a simple proof using the ultrafilter construction) We write $b =< b_\xi >$. Then $b \approx a$ iff for every n the set of indices ξ such that $|b_\xi - a| < 1/n$ is in the ultrafilter. A is open iff for each $a \in A$ there exists some n such that all points within $1/n$ of a are in A. Thus, A is open iff for each $a \in A$ and any $b =< b_\xi >$ near a the set of indices for which $b_\xi \in A$ is in the ultrafilter, and since ${}^*A$ is just the constant sequence in which each entry is the set A in this construction, this is equivalent to the condition $b \in {}^*A$.

ii) (An example of a simple proof using the transfer principle). A is not closed iff there exists a point $p \in \overline{A} - A$ iff $(\forall \varepsilon > 0 \exists a \in A$ such that $\|a - p\| < \varepsilon)$. By transfer this statement is true in the standard world iff it is true in the nonstandard world (with ${}^*A$ replacing A), so iff there exist $a \in {}^*A$ such that $st(a) = p$. Note that we can use standard objects like p in our statements that transfer.

iii) Assuming the Heine-Borel theorem this follows easily from ii). For a proof (due to Robinson) in a general topological space see almost any development of nonstandard analysis, e.g. [2], p. 120. □

Proposition 2. *i) If A is any internal set in the plane then $st(A)$ (the set of all standard points infinitesimally close to points in A) is closed.*

ii) If A is internal, connected and bounded (i.e. contained in some ${}^*B(0, r)$ for a standard real $r > 0$) in the nonstandard plane then $st(A)$ is connected.

Proof. i) If p is a limit point of $st(A)$ then for any standard $\varepsilon > 0$ there must exist points of A within ε of p. Internally we may let $r = \inf_{a \in A} \|a - p\|$, and r must be infinitesimal. But then p is a standard point near points in A, so $p \in st(A)$.

ii). Suppose that $st(A)$ is not connected, and let U and V be a separation, i.e. suppose that $st(A) \subset U \cup V$, both are open, and both have nonempty intersection with $st(A)$. Let $u \in U \cap st(A)$ and $v \in V \cap st(A)$. Then there exists a_u, a_v in A such that $st(a_u) = u$ and $st(a_v) = v$. Since U and V are open, there exists a standard $d > 0$ such that all points within d of u are in U and all points within d of v are in V. Thus, by transfer, $a_u \in {}^*U$, and $a_v \in {}^*V$, and these two sets are open and disjoint. Since A is connected and both ${}^*U$ and ${}^*V$ intersect A, there must exist a point $x \in A$ that is not contained in ${}^*U \cup {}^*V$. Since A is bounded $st(x)$ exists and since $st(x)$ is an element of $st(A)$, it is contained in either U or V. As above, however, this means that x is in either ${}^*U$ or ${}^*V$, and this contradiction completes the proof. □

Here are some properties of continuous functions that illustrate the intuitive nature of nonstandar methods:

Proposition 3. *A standard function f, from $\mathbb{R}^2$ to $\mathbb{R}^2$, is continuous on a domain D iff for all $a, x \in {}^*D$ where a is standard, $(x \approx a) \Rightarrow (f(x) \approx f(a))$. The function f is uniformly continuous on D iff for all $x, y \in {}^*D$ $(x \approx y) \Rightarrow (f(x) \approx f(y))$.*

To clarify the difference between the two definitions, note that a retract from the punctured disk to the boundary, for example, will take two points that are near each other (two points that are both within an infinitesimal of the center) to points that may be on opposite sides of the circle. Thus, continuity alone does not guarantee the stronger condition given in the proposition.

3. More specific examples

In this section we look at results or nonstandard equivalents that are more specific to continuum theory in the plane.

Often the use of nonstandard methods actually allows more simplistic and more intuitive ideas to work. The proof of the standard proposition below is an example.

The proposition is well known and the standard proof is not particularly difficult. However the nonstandard proof below is especially simple. Moreover, the same line of attack does not work, at least easily, in a standard setting.

Proposition 4. *If E is a non-separating plane continuum then for all $\delta > 0$ there exists a set D homeomorphic to the disk such that $E \subset D$ and every point of D is within δ of a point in E (thus E is a countable intersection of sets homeomorphic to the disk).*

Proof. Let $\delta > 0$ be standard and let $\zeta > 0$ be infinitesimal. Let K be an internal finite union of closed ζ- balls that cover E, and let D consist of all points in the nonstandard plane "enclosed" by K, i.e. all points disconnected from infinity by K. We note that $st(K) = E$ since every point of K is within an infinitesimal distance of E and E is compact. There can be no δ-ball contained in D that does not intersect E, for if so the center of such a ball would be a point not in E that is disconnected from infinity by $st(K) = E$ and thus E would disconnect the plane. Thus, in the nonstandard universe there exists such a D with the desired properties, and this is true in the standard universe by transfer. □

If we try to mimic this proof with a standard one in which ζ is simply "small" it is difficult to ensure that much larger balls are not "trapped" inside of K. This proof shows that there do exist sufficiently small standard ζ balls that we could use to construct a standard D in the same way as above, but it is not clear how to make that same argument as simply in a standard version.

The result below gives a kind of "path-connected" equivalent for connectedness in compact sets.

Proposition 5. *A compact set in the plane is connected iff for every two points $p, q \in A$ there exists an internal polygonal path P from p to q such that $st(P) \subseteq A$.*

Proof. Let A be compact and connected. It is easy to see that given any $\varepsilon > 0$ and any points $p, q \in A$ there exists a polygonal path from p to q that stays within ε of A. By transfer there exists an internal polygonal path that stays within an infinitesimal of ${}^*A$ and so has the property that its standard part is contained in A.

Now suppose that A satisfies the condition given but that A is not connected, so that there exist open sets U and V that disconnect A. Choose $p \in U \cap A$ and $q \in V \cap A$, and let P be an internal polygonal path P from p to q such that $st(P) \subseteq A$. Since $p \in U$ but P ends up outside of ${}^*U$ there must exist a point $z \in P \cap \partial({}^*U)$. Since ${}^*U$ is open, $z \notin {}^*U$. Since ${}^*U$ and ${}^*V$ disconnect ${}^*A$ no boundary point of ${}^*U$ is in ${}^*V$. But since U and V are open and their nonstandard counterparts don't contain z, by proposition 1 $st(z)$ is not in either U or V. But $st(z) \in A$, contradicting that A is disconnected by U and V. □

A non-compact set might be connected and not have this property. For example the graph of the $\sin(1/x)$ curve on $(0, 1)$ together with the origin is connected but does not have this property. For more on this see [3].

The proposition below gives a nonstandard characterization of the pseudo-arc. It has the advantage that the condition involves just a single chain in the nonstandard universe.

Proposition 6. *A set A in the plane is a pseudo-arc iff $^*A$ can be covered by an internal chain $\{C_1, ..., C_n\}$ of connected open sets of infinitesimal diameter such that if C_i and C_j are any two links in the chain and B_i and B_j are standard balls centered at $st(C_i)$ and $st(C_j)$ respectively, then the subchain starting at C_i and ending at C_j must include a subchain that intersects B_j and then intersects B_i before it gets to C_j (we note that $st(C_i)$ and $st(C_j)$ are both single points since the diameters are infinitesimal).*

Proof. Suppose that A satisfies the given property. Then by transfer there is a chain of connected open sets that covers A. In the nonstandard universe the infinitesimal chain is a refinement of this chain that satisfies the classic "crookedness" property with respect to it. Thus, again by transfer, in the standard world there is a crooked refinement of the first chain. We can continue in this manner, always using our infinitesimal chain as witness to the existence of a smaller chain that is crooked in any given standard one. In this way we see that A can be written as a nested intersection of chains crooked in the previous one, so that A is a pseudo arc.

Now suppose that A is a pseudo arc, so that it is chainable and hereditarily indecomposable. Since A can be covered by a connected collection of chaining sets of arbitrarily small mesh, by transfer $^*A$ can be covered by such a chaining set in which each link is of diameter less than some infinitesimal δ. Now suppose that this chain does not satisfy the property given in the proposition, so that there exists C_i and C_j in the chain and standard balls B_i and B_j centered at $st(C_i)$ and $st(C_j)$ respectively and that the subchain starting at C_i and ending at C_j does not return to B_i in between intersecting B_j and ending at C_j. We let $X = st(\cup_{k=i}^{j} C_k)$. Then X is compact and connected by proposition 2, so is a subcontinuum of A. We let C_m be the first set in the subchain from C_i to C_j contained in B_j, and let $Y = st(\cup_{k=i}^{m} C_k)$ and $Z = st(\cup_{k=m}^{j} C_k)$ (making suitable index changes if $i > j$). But then $X = Y \cup Z$, and $st(C_i) \notin Z$ and $st(C_j) \notin Y$, contradicting that A is heredetarily indecomposable. □

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