Please answer as many of the following questions as is possible in the time allotted. Although you may not be able to answer all the questions, passing will require you to show *breadth* of knowledge (answer a variety of questions), *depth* of understanding (answer some questions that require explanation) and *ability* to write correct proofs.

Problem 1.

(a) *Give the definition of a* group.

(b) *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be two vector spaces over some field* $\mathbb{K}$.

- *Does the set of all* linear *maps* $L : \mathbf{V} \to \mathbf{W}$ *form a group w.r.t. the operation of* composition*?*
- *If not (and then explain why not), what conditions should be imposed so that we get an example of a (nontrivial) group?*

(c) *Let* $\mathcal{F} = \{F : \mathbb{R} \to \mathbb{R}\}$ *be the set of all* smooth *real-valued functions of a real variable. Is this set a group w.r.t. operations of*

- *composition?*
- *multiplication?*
- *addition?*

In each case, carefully explain why *or* why not.

Problem 2. *Let* G *be a group and* H *a subset of* G.

(a) *Explain what it means for* H *to be a* subgroup *of* G.

(b) *Define a* coset *of* H *in* G.

(c) *Prove that cosets* partition *the group* G *(and make sure to explain what it means). How is this statement related to* Lagrange's Theorem*?*

(d) *Explain why, for any set* S*, a* partition *of* S *is equivalent to defining an* equivalence relation *on* S *(make sure to explain what is an equivalence relation).*

(e) *What is the equivalence relation that corresponds to the partition of* G *by cosets of* H*? Carefully prove that it is indeed an equivalence relation.*

(f) *It is true or false that any partition of* G *corresponds to a partition by cosets of some subgroup* H *of* G*?*

Problem 3.

(a) *Define what it means for two subgroups* H_1 *and* H_2 *of a group* G *to be* conjugated *in* G.

(b) *Show that if* H_1 *and* H_2 *are conjugated in* G*, then they are necessarily isomorphic (and make sure to explain what it means). Is the converse true?*

(c) *Consider the dihedral group* D_4 *of symmetries of a square (this group is sometimes also denoted by* D_8*). Show that the subgroup* H_1 *generated by the reflection about main diagonal and the subgroup* H_2 *generated by the reflection about the horizontal axes of symmetry are* not conjugated *in* D_4*, but* H_2 *and the subgroup* H_3 *generated by the reflection about the vertical axes of symmetry are* conjugated *in* D_4.

(d) *Consider the dihedral group* D_3 *of symmetries of a equilateral triangle. Show that all three subgroups* H_i *generated by reflections* m_i *about the symmetry axes of the triangle are conjugated in* D_3.

(e) *Interpret your conclusions in parts (c) and (d) using (some)* Sylow Theorems *(and make sure to carefully state the necessary theorems and the accompanying definitions).*

Problem 4.

(a) Define an automorphism *of a group G.*

(b) Show that the set $\mathrm{Aut}(G)$ *of all automorphisms of G is a group w.r.t. the operation of composition.*

(c) What is an inner automorphism *of G?*

(d) For $G \simeq \mathbb{Z}_8$, which familiar groups are $\mathrm{Aut}(\mathbb{Z}_8)$ *and* $\mathrm{Inn}(\mathbb{Z}_8)$ *isomorphic to?*

Problem 5.

(a) Define a normal subgroup of a group.

(b) Show that there is a one-to-one correspondence between normal subgroups of a group G and kernels of surjective group homomorphisms from G.

(c) Give an example of a group all of whose subgroups are normal.

(d) Prove that any subgroup $H < G$ of index 2 is normal.

(e) Concrete example maybe?

Problem 6.

(a) Without explicitly listing the elements, determine the cardinality of the group T of rotational symmetries of the regular tetrahedron. ***Hint:*** *use the orbit/stabilizer count.*

(b) Describe the elements of T.

(c) In how many essentially different ways can we color the faces of a regular tetrahedron in three different colors? ***Hint:*** *use Burnside's theorem.*