Collaborative Research: Initiating a Foundational Research Model for Secondary Mathematical Knowledge for Teaching (INFORMS MKT)

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1. Goals and Objectives

We propose foundational research situated in teacher practice to contribute to the understanding of the nature and quality of secondary mathematics teachers' decisions involving their mathematical knowledge for teaching (MKT). As researchers and educators in mathematics teacher education, we share the generally accepted view that teachers' MKT is strongly related to their effectiveness and that more research is needed to advance the field's understanding of how to conceptualize, develop, and assess teachers' MKT (National Resource Council, 2012). Our goal is to understand the nature and role of MKT in expert practice in order to contribute to the development of models of effective MKT and ultimately to inform and improve teacher development. This proposed research addresses the call in the EHR Core Research Program to engage in research on STEM learning environments, such as the interface between teaching and learning. In particular, we focus on how teachers' use of MKT in their work of teaching mediates the quality of students' opportunity for mathematical reasoning.

Our research is grounded in a model of the relationships among teachers' MKT, the work of teaching, and students' mathematical learning (Figure 1). In this model, the work of teaching is decomposed into three interacting areas: planning, instruction, and assessment, where assessment includes both teachers' assessment of student learning and teachers' reflection on instruction. A teacher's MKT influences how these areas are enacted via the use of this knowledge in teachers' decision making. Student learning is influenced by the opportunities for students to engage in mathematical reasoning created by the enactment of planning, instruction, and assessment. Each relationship in our model is bi-directional and heavily mediated by contextual factors. For example, a teacher may have MKT related to a particular topic, but not draw on this knowledge in their planning because the topic is not included in the existing curriculum. With respect to student learning, how a student chooses to engage with an opportunity to reason is a factor that influences their learning.

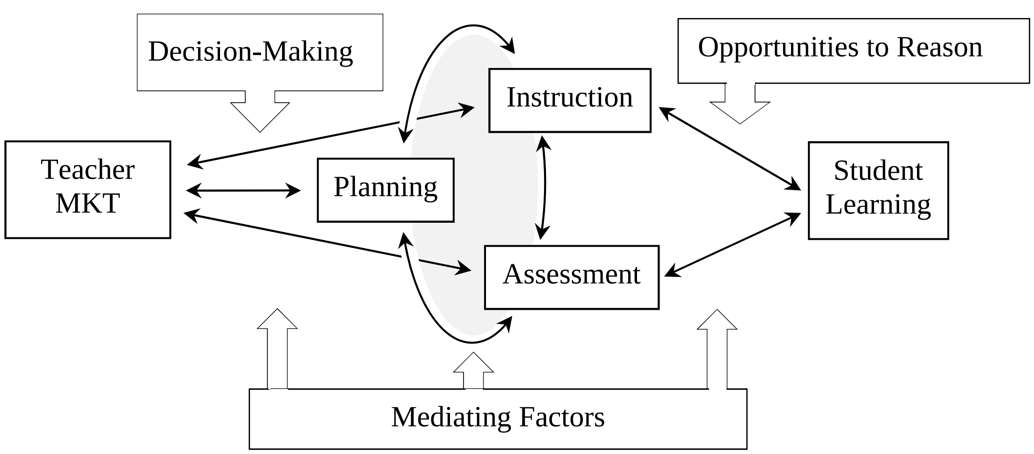

Figure 1. Relationships among teacher knowledge, the work of teaching, and student learning.

No single definition exists for the mathematical knowledge teachers need to effectively teach mathematics, although it is generally accepted that this knowledge consists of knowledge that is purely mathematical and knowledge that is related to the pedagogy of teaching mathematics effectively. For the purposes of our research, we use the term *mathematical knowledge for teaching* (MKT) to encompass both types of knowledge. Further, we use the idea that MKT is the mathematical knowledge, skills, and sensibilities entailed in the work of teaching mathematics to students (Ball, Thames, & Phelps, 2008). We operationalize MKT as the ideas teachers can or do use in the work of teaching that have a mathematical component. Thus, we identify MKT by its mathematical nature and its employment or potential employment in the work of teaching.

Our research questions are:

1. (Enactment) What are the decisions expert teachers make while engaged in the work of teaching exponential functions that involve MKT?
2. (Nature) What is the MKT these teachers draw on while making these decisions?
3. (Mediation) What factors mediate the use of teachers' MKT in their decision making?
4. (Quality) What is the quality of these teachers' decisions with respect to students' opportunity to reason mathematically and how is this quality related to the types of decisions they make, the nature of their MKT, the factors that mediate the decisions?

The research questions and methods for our research are designed to address the following specific needs for MKT research.

Need for research on MKT in teachers' practice. Because MKT is embedded in the work of teaching, there is a need for research focused on teachers' practice (Ball, 2011; Ball, Lubienski, & Mewborn, 2001; Barwell, 2013; Speer & King, 2009; Thompson, 2013). Kajander (2010) suggests that current descriptions of MKT categories provide a general view of MKT, but lack the depth and specificity needed by teachers to change various aspects of their practice. A reason for this is that much current MKT research treats MKT as declarative knowledge rather than one's knowledge to act (Thompson, in press). Similarly, Hashweh (2005) observes, "that in our efforts to understand teacher knowledge and thinking we have focused on knowledge at the expense of thinking processes," which includes how teachers integrate and act on the various aspects of MKT knowledge. Drawing on decision making as the core work of teaching (Shavelson, 1973), we use the lens of decision making to examine MKT in practice.

Need to account for factors mediating teachers' application of their MKT. Kajander (2010) argues that "the teachers' context and viewpoint should remain an important voice in furthering understanding of the domain of mathematics for teaching" (p. 88). Teachers' decisions are strongly influenced by contextual factors such as curriculum resources, school demographics, school leadership, and school policy (Campbell et al., 2014; Petrou & Goulding, 2011). Hill and Charalambous (2012) suspect that contextual factors, such as teachers' perceptions of the need to 'cover' the curriculum, may explain the empirically weak connection they found between teachers' MKT and how teachers involve students in meaning-making and reasoning. Mediating factors, then, render the use of at least some aspects of teachers' knowledge conditional, thus influencing the role of MKT in practice.

Need for content-specific research on MKT. The details of MKT are content-specific (Hashweh, 2005) and prior research on MKT has lacked a focus on the meanings teachers associate with specific mathematics (Thompson, 2013). We therefore focus our research on a specific topic: exponential functions. We chose exponential functions because it is an important topic in the secondary curriculum (Barker & Ganter, 2004; Common Core Standards Writing Team, 2013). In addition, exponential functions are taught in several high school courses (e.g., Algebra II, Pre-Calculus, and College Algebra) which provides more choice of classrooms for research.

Need for focus on the relationship between teachers' MKT and student learning. Scholars, educators, and preliminary research suggest an association between the nature of teachers' MKT and student achievement (Hill, Rowan & Ball, 2005). This link motivates the need to better understand MKT. Tschoshanov (2011) found that teachers who had a better understanding of mathematical concepts and connections tended to have better student achievement and lesson quality. However, the link between teachers' MKT and student learning merits more study. We intend to examine this link by focusing on how teachers' decisions involving MKT influence students' opportunity to reason mathematically. We draw on the premise that "mathematical reasoning is the foundation for the construction of mathematical knowledge" (Ball, Lewis, & Thames, 2008, p. 41). Therefore, students' opportunity to reason should be associated with student learning and thus a worthwhile indicator of the quality of a teacher's decision making with respect to student learning. This analysis can provide insight into the way in which MKT influences student achievement via teachers' decision making.

1.1 Intellectual Merit

Since the articulation by Shulman (1986) that teachers need knowledge related to the content they teach that is particular to teaching, researchers have been working to define and operationalize this idea. In mathematics education, much of this work has focused on describing components of this knowledge (Ball, Thames, & Phelps, 2008; Davis & Simmt, 2006; Rowland, Huckstep, & Thwaites, 2005), assessing teachers' knowledge (Baumert et al., 2010; Hill et al., 2008), and exploring the relationship between this knowledge and student learning (Baumert, 2010; Tchashanov, 2011). Scholars in the field recognize the contributions of this research, but argue that the majority of the research has been conducted outside teachers' practice, thus limiting the development of the field's understanding of how teachers develop and utilize this knowledge. Our proposed research addresses this limitation.

By grounding our research in teachers' practice, we will be able to produce empirical data to advance the field's understanding of the role of MKT in practice. First, we will produce a topic-specific description of the MKT expert teachers are using in their teaching of exponential functions. To date, topic-specific articulations of MKT do not exist, which limits the understanding of what particular knowledge teachers develop and utilize in their practice and what knowledge teacher educators might need to include in their work with teachers. Ideally, our model will serve to launch other work to produce topic-specific MKT descriptions.

Second, we will produce a description of how a teacher's context and environment influences the use of their MKT. Many scholars (Kajander, 2010; Petrou & Goulding, 2011) recognize that context plays an important role in teachers' decision making, but small grain size descriptions, such as the one we will create, do not exist. This type of detail can inform the ways in which productive MKT might be repressed in practice or ways in which teachers can build or maximize the impact of their MKT in practice. Such a perspective adds a dimension to how the field defines MKT. By extending the definition beyond a static knowledge base to include attention to how MKT is enacted in particular contexts, we more accurately mirror how MKT is employed in the work of teaching.

Third, our analysis for evaluating the quality of the decisions teachers make that involve MKT will provide a nuanced picture of the ways that teachers' MKT influences student learning. A key goal of MKT research is to understand how to support teachers to develop MKT that increases their effectiveness in the classroom. By connecting teachers' MKT-based decisions with opportunities for students to reason mathematically, we will be contributing to this understanding.

Another way our research can contribute to the field is by offering methods for studying MKT in practice. Currently, the field does not have established practice-based MKT research strategies. Ball and her colleagues (Ball, 2011) have called for articulating the work of teaching and then identifying the mathematical knowledge needed to carry out that work. Given that the field does not have a well-defined description of the work of teaching, we have chosen to take a related, but different approach to surfacing teachers' MKT in practice. Building on the idea that teaching is the product of making decisions, we have constructed our data collection and analysis methods around identifying the decisions teachers make that involve any mathematical knowledge. This strategy bypasses the need for an a priori classification of the work of teaching, although it might contribute to this classification. Providing the field with an effective method for examining MKT in practice may support further practice-based MKT research. This research strategy is not necessarily specific to mathematics education and may also apply to other content areas, such as science education.

1.2 Broader Impacts of the Proposed Work

Our research will contribute to the knowledge base needed to improve secondary mathematics teacher training by studying the known link between teachers' MKT and teacher effectiveness. We will explore this link by understanding the *enactment, nature, mediating factors* and *quality* of teachers' topic-specific MKT. Our findings will build the knowledge base around MKT by providing ways to further conceptualize MKT and by indicating the nature of the relationship between MKT, teachers' decision making, and opportunities for student learning. These findings can contribute directly to research-based hypotheses about teacher training that supports teachers' development of productive MKT. Further, our research may contribute to methods of evaluating MKT in secondary mathematics teachers.

1.3 Results from Prior NSF Support

Mathematics Teacher Leadership Center (Math TLC): NSF DUE-0832026; $5,379,593; 01/09-06/15
Dr. Novak is PI on the Math TLC, a Math and Science Partnership, which is a collaboration among the University of Northern Colorado (UNC), the University of Wyoming (UW), and school districts throughout Colorado and Wyoming. The project built capacity in teachers, districts, and university faculty and at UNC and UW as well as engaging in research efforts that produced 30 publications. Impacts on student achievement in 6^{th}-12^{th} grade mathematics classrooms are still being studied.

Intellectual Merit. The Math TLC engaged in an extensive research program to understand the impacts of the project on teacher participants and university faculty and to build theory around pedagogical content knowledge. The research program resulted in 30 research publications on topics including:

1) the impact of professional development on teachers'
    a. mathematical understandings (Glassmeyer, 2012B, 2014; Glassmeyer, Oehrtman, & Novak, 2013, 2014)
    b. pedagogical content knowledge (Hauk, Jackson, & Noblet, 2010; Hauk et al., 2013; Hauk et al., 2014; Jackson, Rice, & Noblet, 2011; Powers, Hauk, & Goss, 2013)
    c. perspectives on math teaching and learning (Chamberlin, Goss, Nair, & Breitstein, in press)
    d. cultural competencies for teaching diverse students in Colorado and Wyoming (Hauk, Yestness, & Novak, 2011; Hauk et al., 2012; Parker & Novak, 2013; Powers & Parker, 2013)
2) teacher and faculty responses to and perceptions of blended learning environments (Akarasriworn, 2011; Akarasriworn & Ku, 2013; Akarasriworn & Ku, 2012; Glassmeyer, Rice, & Ku, 2011; Ku et al., 2011A, 2011B, 2011C; Ku et al., 2012; Mayes et al., 2011; Mendoza, Ku, & Akarasriworn, 2012; Powers, Glassmeyer, & Ku, 2011)
3) building community in blended learning environments (Glassmeyer, Dibbs, & Jensen, 2011; Glassmeyer, 2012A; Sachau, 2013)

Broader Impact. The MTLC built a cadre of highly qualified, cultural competent and pedagogically effective secondary mathematics teachers and mathematics teacher leader to improve the math education in 4^{th}-12^{th} grade math classrooms. We also built faculty and university capacity to continue building teacher capacity. In particular, we have the following results which have immediate and longer term impacts that result in improved math education at the 4^{th} – 12^{th} grade levels.

- Development, implementation and institutionalization of a virtual Master of Arts in Mathematics for secondary math teachers that is jointly delivered by UNC and UW. The virtual nature of the program increases access to rural teachers and those who are unable to travel to a UNC or UW to take classes. To date, 53 teachers have received their Masters through this program with another 20 expected to graduate in the next 6 months. We have the capacity to admit a cohort of 18 teachers per year. These teachers can teach high school courses that receive university credit, thus building the capacity of their school to meet the needs of their students.
- Development and implementation of a Mathematics Teacher Leadership Program for 4^{th} – 12^{th} grade math teachers. To date, 44 teachers have participated in this program with 28 of them remaining in or moving into mathematics teacher leadership positions.
- Development of expertise among 7 faculty at UW and 11 at UNC in delivering mathematics and mathematics education courses online

- Creation and dissemination of course materials for the Culture in the Math Classroom (CIMC) course that supports teacher to build their capacity as cultural responsive teachers at the CIMC conference attended by XX mathematics teacher educators

Pathways to Calculus: Disseminating and Scaling a Professional Development Model for Algebra Through Precalculus Teaching and Learning; NSF 1050721; $2.1M; 09/11 – 08/14
Dr. Oehrtman was co-PI on the Pathways to Calculus project.

Intellectual Merit. Pathways to Calculus is a Phase II project was designed to research and address the major barriers to teachers incorporating MKT they had previously developed through participation in a targeted Math and Science Partnership. Since existing curriculum severely limited the teachers' ability to act on their MKT, we authored a research-based pre-calculus curriculum developing these central concepts and reasoning skills (Carlson, Oehrtman, & Moore, 2013). Our research contributed to knowledge of implementation of our professional development model at the K-12, community college, and university levels (Madison et al., accepted; Moore & Carlson, 2012; Moore, 2012).

Broader Impacts. This research contributed knowledge and tools for scaling up the Pathways professional development model and produced insights about factors that contribute to teacher transformation to support students in developing the capacity and confidence to solve novel problems and construct deeper and more connected understanding of the central ideas of a course.

Collaborative Research: Project CLEAR Calculus: Coherent Labs to Enhance Accessible and Rigorous Calculus Instruction; NSF DUE 1245021; $134,218; 07/13 – 06/16
Dr. Oehrtman is PI on Project CLEAR Calculus.

Intellectual Merit. Project CLEAR Calculus is a research-based effort to make calculus conceptually accessible to more students while simultaneously increasing the coherence, rigor, and applicability of the content students are learning. (Oehrtman, Swinyard, & Martin, 2014a; Oehrtman, Swinyard, & Martin, 2014b; Dibbs & Oehrtman, 2014)

Broader Impacts. The project develops quality instruction in introductory calculus sequences, disseminates critical instructor support materials, expands the use of the project labs to successful implementation at other institutions, and assesses student outcomes to characterize the range of variation of prior implementation results and contribute to the broader research knowledge of student learning in calculus.

2. Research Questions, Methods, and Activities

Our research questions are:

1. (Enactment) What are the decisions expert teachers make while engaged in the work of teaching exponential functions that involve MKT?
2. (Nature) What is the MKT these teachers draw on while making these decisions?
3. (Mediation) What factors mediate the use of teachers' MKT in their decision making?
4. (Quality) What is the quality of these teachers' decisions with respect to students' opportunity to reason mathematically and how is this quality related to the types of decisions they make, the nature of their MKT, the factors that mediate the decisions?

2.1 Teacher Recruitment and Selection

We will recruit 12 expert teachers over two years in Algebra II, Pre-Calculus, and College Algebra as these are common high school courses with content related to exponential functions. We draw on Palmer et al. (2005) to identify the characteristics of an *expert* teacher. The expert teacher is someone who is state licensed in secondary mathematics and has taught at least five years and has taught the course under study at least two years. In addition, we will rely on social recognition (reputation) by at least two constituents (administrators, peers, university faculty). We will assess the relevance of the social recognition based on the degree to which the justification provided for a teacher's expertise aligns with Berliner's (2001) 13 prototypic features of teacher expertise. We will also evaluate the degree to which a teacher's district and/or state assessment data provides evidence of consistent growth in student achievement. Finally, we will ascertain if a teacher has additional experience or credentials that suggest expertise, such as having obtained National Board Certification.

We chose to focus on expert teachers for this study because we want to understand the MKT of teachers with a mature practice who have proved to be successful in the classroom, at least with respect to secondary mathematics teachers in general. Practitioners and scholars have long noted the differences between novice and expert teachers; at this point in our research program, we want to develop an understanding of the MKT of teachers who have passed through the novice stage. Future studies might focus on novice teachers to better understand how to develop their MKT.

Each year, we will select a cohort of 6 teachers, 4 from Colorado and 2 from Oklahoma, as we have researchers in both states (Table 1). We will recruit teachers by drawing on the contacts of the research team and by working with officials in the Colorado and Oklahoma State Departments of Education. We will begin by approaching mathematics curriculum coordinators, mathematics coaches, principals or other administrators from several different districts. Our goal is to have teachers from different districts or from different schools within a district to increase the variety of contexts of the teacher participants. Teachers who agree to participate in the study will receive the appropriate informed consent per IRB requirements. In addition, teachers will receive $125 for each lesson observation (we expect 3-5 observations per teacher) and $250 for two additional interviews.

Table 1. Summary of the 12 research participants

Course	**Cohort 1**		**Cohort 2**	
	CO	OK	CO	OK
Algebra II	2	1	2	1
College Algebra or Pre-Calculus	2	1	2	1

2.2 Data Collection

We will collected teacher and student demographics to provide a picture of the population we are studying. Our main data collection efforts will be around documenting teachers' practice.

Teacher and Student Demographics

Once the selected teachers have agreed to participate in the study, we will collect demographic information on the teacher participants and their students. We will collect the teachers' gender, race/ethnicity, secondary mathematics certification status, highest degree earned, other teaching certifications, number of years teaching, number of years teaching secondary (7-12) mathematics, number of years teaching in their current district, and years teaching the current course. We will collect students' gender, race/ethnicity, special needs identification, English language learner identification, socioeconomic status indicator, age, and whether the student is repeating the course. These data will be used to provide context for our data analysis and findings. Data will be sanitized and secured per IRB requirements.

Data from Teacher Practice

We will gather data on multiple aspects of each teacher's practice related to teaching exponential functions. Based on available district planning documents, we expect each teacher will spend 3-5 days of instruction on exponential functions in a given course.

For each lesson, we will collect data as follows:

- Pre-observation interview (60-90 minutes each)
- Class observation (~60 minutes each)
- Post-observation interview (60-90 minutes each)

Following all of the classroom observations, we will conduct two additional interviews

- Final observation interview (60-90 minutes)
- MKT interview (60-90 minutes)

We have generated detailed protocols for each of the types of interviews and observations that we will conduct with teachers. We have also developed an overarching approach we call our *general interview protocol*. This general protocol serves the purpose of aligning our interview methods with our theoretical perspective and data analysis methods. Our underlying premise is that teachers use MKT to make decisions in their work of teaching. Therefore, we are interested in understanding what decisions teachers are making in their planning, instruction, and assessment and the rationale for those decisions. Decision rationale that relies on any mathematical knowledge or thinking is what we identify as enactment of MKT. Our general interview protocol, then, is based on surfacing teacher decisions and the rationale for those decisions, with a particular goal of determining any mathematical thinking embedded in the decision rationale.

Another key element in our theoretical perspective is that contextual factors mediate teachers' use of their MKT. Therefore, we want to probe the teachers' decision making to understand how context may have influenced the rationale for those decisions. We recognize that district policies, student characteristics, and curriculum are likely to be considered by the teachers, but will work to identify other contextual factors as well.

The design of our specific interview protocols (e.g., pre-observation interview) is based on considering what things teachers are likely to be making decisions about in the areas we are interviewing them about. For example, in the pre-observation interview, we are interested in teachers' planning of instruction. We posit that teachers need to make several key decisions: what the learning goals are, how to structure the lesson activities, and how to assess student learning during the lesson. Thus, we structure the interview questions to surface what decisions teachers have made with regard these planning activities.

All interviews will be semi-structured in that we will need to ask specific follow-up questions based on teacher responses to our primary questions. One consideration for asking follow-up questions is to minimize any influence we have on teachers' thinking during the pre- and post-observation interviews. That is, we want to avoid asking questions that inserts MKT into the conversation. For example, we want to avoid questions such as "How will you make sure students understand exponential growth is multiplicative in nature?" if the teacher has not previously brought up this concept. This guideline will be relaxed for the Final Observation Interview as this interview is conducted after the class observations are completed and its purpose is to probe more deeply, if we need to, to better understand a teacher's thinking.

In order to maximize the quality of the interview data we collect, we will have at least one other researcher observe the interview remotely. The observing researcher can suggest probing and clarifying questions via text chat, which will increase the quality and consistency of the questioning to surface teachers' MKT.

Interviews will be recorded and any artifacts, such as lesson plans and handouts, will be collected. The pre-observation interview will focus on the rationale for the instructional plan. During the lesson, the class will be video-recorded and a researcher will take field notes. The video camera will be manually operated to follow the teacher who will wear a microphone. The researcher taking field notes will document times during the lesson that appear to represent key decision making by the teacher that can then be examined in post-observation video analysis and post-observation teacher interviews. We will also collect any relevant artifacts, such as student work. For the post-observation interview, we will focus on the rationale for the teacher's evaluation of the lesson and how they plan to adjust future instruction accordingly.

After we have finished collecting data from the set of classes focused on exponential functions, we will conduct two additional interviews. The first, which we call the Final Observation Interview, will focus on probing for additional information related to the rationale for teachers' decisions. We anticipate occasionally being unable to probe as deeply as we would like during the pre- and post-observation interviews because of 1) time limitations or 2) probing more deeply would likely influence a teacher's current MKT. For information related to the delivery of the lesson, we may show the teacher video clips from the lesson and ask them to explain their decision-making for the episodes shown in the video.

The second post-observation interview will be the MKT Interview, which will be common to all the teachers. In this interview, we will pose three instructional scenarios to teachers and ask them to consider how they would address the issues in each scenario. The math content in each scenario will be an important idea related to understanding exponential functions based on existing research literature. Each scenario will have a different pedagogical component with one scenario being in the context of planning for instruction, a second scenario being in the context of teaching a class, and a third scenario being in the context of assessing student work.

2.3 Data Analysis

Our data analysis will have three key phases. In the first phase, we analyze the interview and observation data related to the teachers' instruction of exponential functions. This analysis is designed to address our first three research questions. In the second phase, we analyze the MKT Interview data. This analysis supplements the findings with respect to research questions 2 and 3. Lastly, we conduct the analysis to address research question 4, which involves evaluating the quality of teachers' decisions. Throughout the data analysis phases, multiple researchers will code and interpret the data to refine and calibrate our analysis methods.

Interview and Observation Data Analysis

The first step in this phase of the data analysis is to extract from these data the decisions teachers made in their work of teaching exponential functions that involved some form of mathematical knowledge. Table 2 below shows an example analysis drawn from pilot data we collected. Due to space limitations, we provide only summaries for each entry. Specifically, we will identify the teaching activity during which the decision was made (planning, instruction, assessment), what decision was made, the mathematical knowledge associated with the decision, and the rational and context. Our interview protocols are designed to elicit these considerations from teachers.

Table 2. Sample summaries from the pilot data.

Activity	Decision	Mathematical Knowledge	Rationale and Context
Planning	A goal for the lesson is for students to be able to look at a story problem and identify whether it can be modeled with a linear or exponential function	Information in story problems can be used to determine if an exponential function is an appropriate function to model the situation in the problem	The standards call for students to be able to determine when a contextualized situation is linear or exponential Students had not experienced this kind of practice and were in the habit of following procedures rather than interpreting story problems
Planning	The lesson will begin by reviewing the characteristics of linear and exponential functions	Characteristics of linear and exponential functions can be compared in terms of their equations, graphs, and tables	Students have already had exposure to these elements of both linear and exponential functions
Instruction	Teacher says, "linear functions are about addition and exponential functions are about multiplication"	Linear functions have a "starting value" (the y-intercept, $f(0)$, or constant term) and a constant amount added for each unit increase in the domain (the x-coefficient or slope). Exponential functions have a starting value (the y-intercept, $f(0)$, or constant factor) and a constant multiple change for each unit increase in the domain (the base of the exponent).	Teacher believes this information can be effectively summarized for the students as 'linear functions are about addition and exponential functions are about multiplication'.

Activity	Decision	Mathematical Knowledge	Rationale and Context
Instruction	Teacher accepts the response “Exponential function because of the word ‘triple.’” to her question of “What type of function is problem 1c?”	The word ‘triple’ in a word problem indicates multiplicative change, indicating the word problem represents an exponential function	Students are working on problems that are known to model either exponential or linear functions
Instruction	Teacher response to the student question “What is *x*?” is: “We don’t know.”	The exponent x in the equation $y = a \cdot b^x$ is a variable and so does not represent a specific number	Student posed the question “What is *x*?” when the class is working on finding the equation for a word problem that represents an exponential situation (worksheet problem 1c)
Assessment (reflection)	The discussion in class about understanding the exponential growth of dividing by 2 is the same as multiplying by ½ went well.	The growth factor in exponential functions can be thought of as multiplying or dividing by a number. However, division can also be interpreted as multiplication by using the reciprocal of the number being used for division.	Teacher believes it’s important for students to be able to think about all 4 arithmetic operations in terms of addition and multiplication with subtraction and division being inverse operations of addition and division because in her experience mathematics most often emphasizes addition and multiplication. For example, we think of multiplying by ½ instead of dividing by 2.
Assessment (reflection)	It was fine for students to work only with ‘well-ordered’ tables when determining if a function is exponential and examine growth only in terms of changes in *y*.	Both the *x* and *y* columns in a table are needed to ascertain the growth pattern in a function, but in ‘well ordered’ tables (the *x* value starts at zero and increases by 1) you only have to look at the patterns in the *y* column	Teacher believes that the students in this class might be overwhelmed by working with tables that are not ‘well ordered’ as that is ‘tricky’, so she will have them work with ‘well ordered’ tables for now so they don’t have to pay attention to the *x* column in tables.

Our goal is to conduct this initial analysis from the lesson observation data prior to the Post-Observation Interview so that we might better identify the teacher’s rationale for decisions made during instruction. We will be able to use video clips to help the teacher recall specific instances during instruction. In addition, we will use the Post-Observation Interview to further supplement our data with respect to a teacher’s rationale.

Once the data extraction and organization has been completed, we will be able to address our first research question by categorizing the types of decisions teachers made and to associate these decision types within the different teaching activities of planning, instruction, and assessment. We will describe patterns in teachers’ decision types for individual teachers as well as compare these patterns across the teachers. These descriptions provide an overview of the role of MKT in teachers’ practice as it represents the particular work of teaching in which teachers draw on their MKT. This can inform the particular aspects of teaching most influenced by MKT in practice.

Our second research question involves creating descriptions of each teacher’s MKT. To do this, we will organize the decision data around the mathematical knowledge associated with that decision. Then, we will have a set of decisions associated with each type of mathematical knowledge that we can use to describe teachers’ pedagogical ideas associated with the math knowledge. Once this detailed description is created, we will explore meaningful ways to represent this description that are accessible and illuminate key threads of understanding.

To address our third research question, we will examine the role of mediating factors in teachers' decisions. We define a mediating factor as information or an idea the teacher employs in making a decision in the work of teaching that appears to modify the nature of the teachers' MKT in its application. For instance, a teacher may know of a line of questioning that can support students' development of a concept, but choose not to pursue this line of questioning because they do not believe it will be assessed. This analysis is likely to surface not only the contextual factors that teachers pay attention to in their decision making, but also their underlying beliefs about mathematics, teaching, and learning.

MKT Interview Data Analysis

The data from the MKT Interview will be used for two purposes. First, it will be used to support the development of the descriptions of the teachers' MKT. Because this interview is conducted outside the immediacy of a teacher's practice, we can explore more deeply the teacher's meanings, images, and ways of thinking about exponential functions, which can be used to supplement the description of the teacher's MKT that emerged from the observations and interviews related to specific lessons. Second, data from this interview can provide a picture of teachers' decision making outside of the immediacy of practice that might serve to highlight ways in which contextual factors influence teachers' decision making in practice. This juxtaposition could also inform how teachers talk about their decision making outside their practice versus how they actually make decisions. The results from this analysis might have implications for the assessment of teachers' MKT.

Data Analysis for Quality

This stage of the data analysis is intended to address the last research question: What is the quality of these teachers' decisions with respect to students' opportunity to reason mathematically and how is this quality related to the types of decisions they make, the nature of their MKT, the factors that mediate the decisions?

The foundation for conducting this analysis is identifying a set of ideas students should have the opportunity to reason about with respect learning exponential functions at the high school level. This will provide a map against which to compare students' opportunity to reason within and across the different classrooms. We have already created an initial conceptual framework for exponential functions that we used as a basis for creating an assessment on high school level understanding of exponential functions. The framework creation was guided by the literature and the Common Core State Standards for Mathematics. The framework decomposes aspects of exponential functions along multiple dimensions:

- mathematical structure
    - splitting (Confrey and Smith, 1994, 1995; Confrey, 1988; Smith, Haarer, & Confrey, 1997)
    - partial successors (Confrey & Smith, 1995; Strom, 2008)
    - rate proportional to amount (Carlson & Oehrtman, 2011))
- reasoning modes (e.g., pre-functional, correspondence, covariation (Thompson, 2008))
- representation (e.g., algebraic, graphical, numeric, contextual)
- levels of understanding using a learning trajectory (Ellis et al., 2013) and the PARCC performance level descriptors (PARCC, 2012)).

We will use the classroom observation data to identify places during instruction where students had the opportunity to reason about mathematics in general and exponential functions in particular. We will decompose these opportunities into content and cognitive demand components (Marzano, 2009). The cognitive demand component will identify the reasoning level of the opportunity using a hierarchy identifying lower and higher demand levels. For example, a teacher may pose a question to the class for the students' consideration that is related to graphical representations of exponential functions (content) that could be lower cognitive demand (recognition) or higher cognitive demand (justification). As a result of this analysis, we will be able to describe the ways in which the opportunities for students to reason varied across the classrooms as well as the degree to which these opportunities to reason aligned with our conceptual framework.

Once we have produced descriptions of the opportunities to reason for each teacher, we look for any relationships that exist between teachers' decision making and the nature of the students' opportunities to learn. For this we will draw on our previous analysis of the types of decisions teachers make that involve MKT, the descriptions of teachers' MKT, and the contextual factors that influenced the use of teachers' MKT. Using these data we will generate hypotheses about possible relationships and then use constant comparative techniques to validate, refine, or reject these hypotheses.

While we are not explicitly studying the relationship between teachers' MKT and a measure of student learning, we will be giving a pre- and post-assessment on exponential functions to students. We have been working on developing and validating this assessment for two years. Despite a small sample size (N = 12), we want to perform preliminary quantitative data analysis to i) determine if the hypothesis is reasonable that richer MKT of teachers as measured by the quality of the opportunities to reason is associated with greater mathematical knowledge and skills of their students, and ii) to set a foundation for future research exploring this link. To that end, we propose to categorize teachers based on the richness of their MKT. Our plan is to create two groups of teachers, a high (richer) group and a low (less rich) group. Then we will perform an analysis of covariance with our two categories of teachers as the independent variable, students' post-assessment scores of the exponential functions assessment as the dependent variable, and students' pre-assessment scores of the exponential functions assessment as the covariate. This analysis will test whether there is a statistically significant difference between teachers with richer MKT and teachers with less rich MKT on students' post-assessment scores controlling for pre-assessment scores. Because these results would be preliminary, they are not generalizable but rather would be used as a basis for further study.

3. Project Management Plan

In this section, we outline the roles and responsibilities of members of the INFORMS MKT research team, provide a project implementation schedule and address potential risks associated with the research.

3.1 Personnel Roles and Responsibilities

The roles and responsibilities of INFORMS MKT team members are outlined below.

M. Oehrtman (PI): Leads the research effort; supervises OSU graduate student research; recruits Oklahoma participants; liaison with school districts of Oklahoma participants; responsible for data collection in Oklahoma; liaison with Dr. Ellis, external evaluator; liaison with NSF program officer.

J. Novak (PI, PD): Oversees and coordinates all project activities; recruits Colorado participants; liaison with school districts of Colorado participants; contributes to data collection, analysis and dissemination; oversees NSF reporting.

F. Parker (co-PI): Contributes to the research program (data collection, analysis and dissemination); responsible for data collection in Colorado; liaison with Dr. Thames, external evaluator; supervises UNC graduate student research; oversees data management;

R. Powers (co-PI): Oversees development, administration and analysis of the exponential functions assessment; works with school district personnel to gain approval for engaging in research in their district and access to needed district-wide data; contributes to the research program (data collection, analysis and dissemination).

External evaluators (A. Ellis, M. Thames): Provide formative evaluation at key points for continuous improvement of our research.

For this collaborative proposal, the personnel from OSU and UNC will work closely with each other, making significant use of technology (online meetings, google docs, online communication and data management tools) to conduct weekly project meetings. While data is being collected in two states, the data will be analyzed by all project personnel. To support intensive project work during data analysis, project personnel will hold face-to-face meetings every 3-4 months. For external evaluation, we will make similar use of technology as well as conducting face-to-face meetings as appropriate.

3.2 Project Implementation Schedule

Table 3 contains a timeline of the research activities which we detailed in the Data Collection and Analysis section and the evaluation activities which we detail in the Project Evaluation section. Before the project begins, we will submit an IRB for institutional approval and validate the exponential functions assessment. The major project activities during the first two years are collecting and analyzing data for two cohorts of teachers. Towards the end of year two, we begin cross-cohort analysis which continues in year 3. Throughout the project we engage in evaluation activities, dissemination of our findings, and documentation of the evolution of our research methods for dissemination.

Disseminating of Research Findings and Methods

We will disseminate the preliminary and cumulative results of our research through submissions to peer-reviewed journals and high-profile conferences targeted at mathematics teacher educators and researchers. Conference presentations, proceedings, and peer-reviewed journal articles provide avenues to present findings related to our characterization of the *enactment, nature, mediating factors* and *quality* of expert teachers' MKT while teaching exponential functions and to our exponential functions assessment.

To realize the capacity-expanding potential of our work for basic education research, we document our extensive research methods for the research community to use and adapt. In addition, we document how our methods evolved over the duration of the project so users know why we made the choices we did. This documentation lays the foundation for research on secondary and tertiary MKT that will persist beyond the duration of the funded project and generate results beyond our initial content focus on exponential functions.

Data Sharing

At the end of the project, we will submit all data allowable under our IRB approval to the Inter-university Consortium for Political and Social Research (ICPSR) for sharing with other researchers. ICPSR will make the data available to the broader social science research community. All data, including video, generated during the project will be restricted-use since removing potentially identifying information would significantly impair the analytic potential of the data. Users (and their institutions) must apply for these files, create data security plans, and agree to other access controls.

Table 3. Timeline of project activities.

Time Frame	**Activity Description**
	Year 1: August 2015 through July 2016
Aug	External review of data collection protocols
Aug – Sept	Recruit teachers for Cohort 1
Sept	Revise and calibrate data collection protocols
Oct – May	Collect data for Cohort 1
Oct – May	Analyze Cohort 1 data
Nov, March, July	Document the evolution of our research methods
March – May	Recruit teachers for Cohort 2
May	Analyze exponential functions assessment data
May – July	Prepare initial findings report on the enactment, nature, mediating factors and quality of expert teachers' MKT while teaching exponential functions
June – July	Prepare NSF annual report
June	External review of revised data collection protocols
July	Revise data collection protocols Revise exponential functions assessment, if needed
	Year 2: August 2016 through July 2017
Aug	External review of our initial documentation of our research methods
Aug-Sept	External review of initial findings report on the enactment, nature, mediating factors and quality of expert teachers' MKT while teaching exponential functions
Aug – May	Collect data for Cohort 2
Oct	Revise data analysis protocols based one external review
Nov, March, July	Document the evolution of our research methods

Time Frame	Activity Description
Nov – April	Present preliminary findings at conferences (AMTE, NCTM research pre-session, PMA-NA, CRUME)
Nov – May	Analyze Cohort 2 data
May – July	Conduct cross-cohort analysis
May – July	Analyze exponential functions assessment data Draft research report on the exponential functions assessment
June – July	Prepare NSF annual report
Year 3: August 2017 through July 2018	
Aug	External review of our updated documentation of our research methods
Aug – Sept	External review of research report on exponential functions assessment
Aug – Oct	Continue cross-cohort analysis
Oct – Nov	Prepare final research report on exponential functions assessment for journal submission
Nov – Feb	Draft research report(s) on our characterization of the *enactment, nature, mediating factors and quality* of expert teachers' MKT while teaching exponential functions
Nov – Feb	Finalize documentation of our research methods and their evolution
Nov – April	Present findings at conferences (AMTE, NCTM research pre-session, PMA-NA, CRUME)
March – April	External review of draft research report(s) on our characterization of the *enactment, nature, mediating factors and quality* of expert teachers' MKT while teaching exponential functions External review of documentation of our research methods and their evolution
May – July	Prepare final research report(s) on our characterization of the *enactment, nature, mediating factors and quality* of expert teachers' MKT while teaching exponential functions for journal submission Prepare final documentation of our research methods and their evolution for dissemination
May – July	Prepare data for storage with ICPSR
June – July	Prepare NSF annual and outcomes report

3.3 Project Risks

Here we outline our strategy for dealing with some possible risks associated with our research design.

Recruiting Risks and Mitigation Plan

The primary risk in the recruitment process is that districts will refuse to allow us to study their teachers. School districts, reasonably so, seek to protect district-wide data, limit district cost to gather district-wide data for the project, limit intrusions into teachers' classrooms, and ensure that they authorize only ethical, well-planned research in their district. Each of these items represents a potential barrier for authorizing the participation of a teacher in our study.

When we approach districts, we will already have IRB approval from our institution as indicator of ethical, well-planned research. We will, at a district's request, go through any internal IRB or research review process they have which serves both to protect district-wide data and to ensure ethical, well-planned research. We will provide compensation at $1000 per teacher to the district for any time needed to collect district-wide data; we can increase this amount, if needed. We will be observing 3-5 class sessions for each participating teacher which we will do as unobtrusively as possible. Teacher interviews will take place outside of contract time to ensure teachers will not miss class to participate in the research.

Attrition Risk and Mitigation Plan

As with any research study involving human participants, there is always a risk of participant attrition. The short time frame of 3-4 weeks when teachers need to be available for research reduces the likelihood of attrition. The short research time frame within a 9-month school year combined with the fact that exponential functions are taught at a variety of times across the school year means that should a teacher drop out of the research before or during the data collection, we will have time to study another teacher. To ensure there is another teacher to study, we will over recruit teachers (1 extra in Oklahoma and 2 extra in Colorado each year) within the districts we are working with. In the unlikely event that we do not have enough back up teachers, we will first look for expert teachers in our current districts. If no appropriate teacher is available, we will begin working with another district to identify teacher participants.

We are providing a financial incentive (up to $875 per teacher) to compensate teachers for their time outside of contract time needed for the interviews. The financial incentive is likely to reduce attrition. If the financial incentive is viewed as too low at the district level or by individual teachers, we can increase the financial incentive appropriately for all teachers.

4. Project Evaluation

We seek external, objective evaluation to provide ongoing, critical review of our research design and activities, including our theoretical framework, data collection protocols, analyses, and reporting (Sutton & Callow-Heusser, 2014) to ensure that we are engaging in high quality research that is appropriately informed by the literature and informing the research community (Guthrie et al., 2013). Two external evaluators with expertise in the field have already agreed to serve: Dr. Mark Thames (University of Michigan) and Dr. Amy Ellis (University of Wisconsin-Madison). We will engage other external reviewers and consultants, as needed, with expertise in mathematical knowledge for teaching, the learning and teaching of exponential functions, and observing and documenting math classrooms to evaluate and provide formative feedback on our progress.

Dr. Thames researches mathematics teaching and contributes to the training of mathematics teachers. His research investigates the practice of teaching mathematics and the improvement of practice. He investigates the work entailed in teaching, the mathematical demands of that work, and ways in which these contribute to learning to teach. He is interested specifically in equitable teaching practice, measures of teacher knowledge and practice, and designs for collective work on teaching.

Dr. Ellis conducts research on students' learning, including quantitative, proportional, functional, and algebraic reasoning, with consistent emphasis on the meanings that individuals construct and the ways of reasoning those meanings support. Within this research she developed learning progressions for exponential functions (Ellis et al., 2013) and will contribute her expertise to guide and evaluate the central role of content-specific nature of MKT and its impact on decisions teachers make about their practice.

We plan to engage our external reviewers at keys points in the research throughout the life of the project. See Table 4. For each of the reviews in Table 4, we ask the evaluator to write a report answering these evaluation questions (Mattessich 2012):

1. To what extent is the project team engaging in high-quality research?
2. In what ways can the project team increase the quality of the research?

These reports serve as formative evaluation of our project. In addition to the reports, we anticipate discussing the review with the reviewers during face-to-face meetings, online meetings, and phone calls to gain the most benefit from the analysis. Excerpts of the reports will be included in our annual reports.

Table 4. External review timeline, task and data for the task.

Timeline	External Evaluation Task	Data for Task
Aug 2015	Review data collection protocols	Pre-observation interview protocol, observation protocol, post-observation protocol, final observation interview protocol, MKT interview protocol
June 2016	Review revised data collection protocols	Revised pre-observation interview protocol, revised observation protocol, revised post-observation protocol, revised final observation interview protocol, revised MKT interview protocol
Aug 2016	Review documentation of our research methods	Data collection protocols, data analysis protocols, initial report of the evolution of our research methods
Aug – Sept 2016	Review initial findings report on the enactment, nature, mediating factors and quality of expert teachers’ MKT while teaching	Representative samples of our data analysis, initial findings report

Timeline	External Evaluation Task	Data for Task
Aug 2017	Review documentation of our research methods	Data collection protocols, data analysis protocols, updated report of the evolution of our research methods
Aug – Sept 2017	Review research report on exponential functions assessment	Revised exponential functions assessment, reliability and validity studies, research report
March – April 2018	External review of draft research report(s) on the *enactment, nature, mediating factors and quality* of expert teachers' MKT while teaching exponential functions	Draft research report on the *enactment, nature, mediating factors and quality* of expert teachers' MKT while teaching exponential functions
March – April 2018	External review of documentation of our research methods and their evolution	Draft report documenting our research methods and their evolution