Research Description

Anton Dzhamay

Introduction

My research is in the fields of *integrable hierarchies* and *soliton equations*, *WDVV* (or *associativity*) equations and *Frobenius manifolds*. I will first give a brief overview of those subjects, paying special attention to the relationships between them, and then describe my previous results and projects I am currently working on.

Soliton Equations and Integrable Hierarchies The origins of the theory of *soliton equations* can be traced back to the month of August 1834, when John Scott Russell first observed the curious phenomena of the *Great Wave of Translation* (or the *Great Solitary Wave*) on the English–Glasgow canal [Rus44]:

> "...I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped — not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary evolution, a rounded, smooth, and well-defined heap of water, which continued its course along the channel apparently without change of form or dimunition of speed...".

But although mathematical description of this phenomena was studied by de Boussinesq [dB72], Rayleigh and others in the end of the XIX century, and the famous *KdV equation* describing the propagation of waves on the surface of a shallow channel was obtained by Kortweg and de Vries in 1895 [KdV95], there was little progress made in understanding the nature of the *solitary wave* until 1966 when, in the attempt to understand the results of the famous Fermi–Pasta–Ulam experiment [FPU74], C. Gardner, J. Green, M. Kruskal, and R. Miura found the *Inverse Scattering Transform* method for integrating the KdV equation [GGKM67] and the theory of soliton equations was born. Since then, many important equations of mathematical physics (*Kadomtsev–Petviashvili or KP, Boussinesq, sine-Gordon, Toda Lattice, Non-Linear Schrödinger or NLS*, and others) were represented in the soliton framework. Moreover, each of these classical equations is usually just a first member in the infinite tower of equations called the *hierarchy*. Soliton equations are remarkable in many aspects. For example, they have *infinitely many* conserved quantities (*integrals of motion*), and thus are examples of *completely integrable systems*. These equations are non-linear, but it is still possible to find exact analytic solutions. Some of these solutions, the so-called N*-solitons*, look like localized waves and satisfy a non-linear superposition principle (or non-elastic scattering) — when two such solutions meet, they undergo a complicated non-linear interaction, but emerge from it undamaged, i.e., they behave both like waves and like particles (this property of solitons motivated their use in fiber-optic communications). To me the most interesting aspect is the ubiquity of soliton equations and hidden relationships between many different branches of science, from classical differential geometry of the XIX century to latest advances in modern theoretical physics, that they help to uncover. I am particularly interested in the relationship between soliton equations and *algebraic geometry*. This relationship stems from the fact that soliton equations can be represented as a *compatibility conditions for auxiliary overdetermined linear systems*. This property of soliton equations was discovered by P. Lax [Lax68] and is now called the *Lax representation*. The main ideas of the algebro-geometric method were developed in a series of papers by S. Novikov [Nov74], B. Dubrovin [DN74], V. Matveev, A. Its [IM75], I. Krichever [Kri77], H. McKean, and P. van Moerbeke[MvM75] and can be roughly described as follows. One constructs special kind of functions Ψ_{BA}, called the *Baker-Akhiezer functions*, whose properties ensure that they are solutions for the auxiliary linear problems of the Lax form. These linear problems are then compatible (since Ψ_{BA} is a solution), and so one obtains solutions to the corresponding soliton equations. Baker-Akhiezer functions are constructed by means of algebraic geometry. One starts with a Riemann surface Γ of genus g and a collection of n points on Γ (together with local coordinates around these points). Out of this *algebro-geometric data* one constructs meromorphic functions on Γ that have prescribed essential singularities at selected points — the Baker-Akhiezer functions. The resulting formulas for the solutions of the soliton equations are known as the *theta-function* formulas, since they are expressed in terms of Riemann θ-function. Note that we in fact get a relationship between the *moduli space of Riemann surfaces* $\mathcal{M}_{g,n}$ with some extra data and a *moduli space of periodic and quasi-periodic solutions of soliton equations*. This relationship turned out to be very deep and it had a lot of impact on algebraic geometry. I shall mention two famous examples.

Riemann–Schottky Problem and Novikov's conjecture It is well-known that not every Riemann θ-function is associated with a Riemann surface. The Schottky problem is a problem of determining if a given θ-function actually *is* coming from a Riemann surface. Novikov suggested that this happens if and only if it produces solutions to the KP equation! This amazing result was later proved by T. Shiota [Shi86].

Witten's Conjecture Moduli spaces of curves $\mathcal{M}_g$, $\mathcal{M}_{g,n}$, and the like are important objects of study in classical algebraic geometry. Recent developments in *String Theory* and *Quantum Field Theory* attracted to these spaces the attention of physicists. Based on the ideas from the theory of two-dimensional quantum gravity, E. Witten suggested in [Wit91] that the intersection theory on the moduli space $\mathcal{M}_{g,n}$ (such theory carries a lot of information about the structure of $\mathcal{M}_{g,n}$) is completely determined by the τ-function of a KdV hierarchy. This conjecture was later proved by M. Kontsevich [Kon92].

Whitham Hierarchies Recently a new type of integrable hierarchies began moving into the spotlight. These hierarchies can be thought of as quasi-classical limits of classical integrable hierarchies [TT95], and are called *dispersionless* or *Whitham* hierarchies. Equations of the hierarchy are called *Whitham* (or modulation) equations. From the algebro-geometric point of view these equations appear in the theory of perturbations of exact algebro-geometric solutions of soliton equations by the non-linear WKB (or Whitham averaging) method and describe the "slow drift" (Whitham flow) on the moduli space of algebro-geometric data necessary to ensure the solvability of the perturbed problem by means of asymptotic expansion. These equations have a certain "universal" structure and can be described just in terms of algebro-geometric data: each Whitham flow on the moduli can be generated by a corresponding differentials $d\Omega_i$ on the universal curve; this approach to Whitham hierarchies was suggested in [Kri94]. Whitham equations began to attract more attention recently in view of their connection to the *Seiberg-Witten equations* of the string theory [BK00] and to the WDVV equations of the topological field theory.

WDVV equations and the Frobenius manifolds WDVV or associativity equations appeared in the work of E. Witten, R. Dijkgraaf, E. Verlinde, and H. Verlinde ([DVV91b, DVV91a, Wit90] on the deformations of $2D$ topological field theories (TFT). One can show that the structure of TFT is equivalent to a structure of a *Frobenius algebra* A, i.e., a commutative associative algebra with a unit and a symmetric non-degenerate bilinear form η such that $\eta(a * b, d) = \eta(a, b * d)$. Let $\{\phi_\alpha\}$ be a basis of A (ϕ_α correspond to primary fields in TFT) and let $\eta_{\alpha\beta} = \eta(\phi_\alpha, \phi_\beta)$, $c_{\alpha\beta\gamma} = \eta(\phi_\alpha, \phi_\beta * \phi_\gamma)$. Consider deformations of this structure coupled to the primary fields: to each ϕ_α corresponds a deformation parameter t_α. If there exists a function $\mathcal{F}(\mathbf{t})$ such that $\eta_{\alpha\beta} = \partial_{0\alpha\beta}\mathcal{F}(\mathbf{t})$ remains constant and $c_{\alpha\beta\gamma}(\mathbf{t}) = \partial_{\alpha\beta\gamma}\mathcal{F}(\mathbf{t})$, then this deformation is called potential and the function $\mathcal{F}$ is called the *WDVV potential*. Then the WDVV equations is the following overdetermined non-linear system of partial differential equations expressing the associativity conditions for the deformed algebra structure:

$$\mathcal{F}_{\alpha\beta\lambda}(\mathcal{F}_{0\lambda\mu})^{-1}\mathcal{F}_{\mu\gamma\delta} = \mathcal{F}_{\delta\beta\lambda}(\mathcal{F}_{0\lambda\mu})^{-1}\mathcal{F}_{\mu\gamma\alpha}. \qquad \text{(WDVV)}$$

Geometric and coordinate-free approach to the theory of the WDVV equations was introduced by B. Dubrovin in [Dub96a] and is now called a theory of *Frobenius manifolds*. Frobenius manifolds have a very rich structure and are related to many diverse branches of mathematics, from the very classical ones (Darboux-Egoroff metrics, n-orthogonal curvilinear coordinate systems, unfolding of singularities, Painlevé equations) to the very modern (quantum cohomology, Gromov-Witten invariants, integrable hierarchies, the Seiberg-Witten equations). A large class of solutions to the WDVV equations corresponding to the Landau-Ginzburg theories and minimal models can be obtained from the theory of Whitham hierarchies. For a very simple and beautiful example of the A_n minimal model, a Frobenius algebra structure is defined on the space of degree $n-2$ polynomials in p with the help of the so-called superpotential $W(p) = \dfrac{p^n}{n}$ by

$$u * v = u(p)v(p) \mod W'(p),$$
$$\langle uv \rangle = -\operatorname{res}_\infty \frac{u(p)v(p)}{W'(p)} dp.$$

A Frobenius algebra structure is then deformed by deforming the superpotential W. The closed expression for the corresponding WDVV potential $\mathcal{F}(t)$ was identified by I. Krichever [Kri92] with the logarithm of the τ-function of a certain reduction of the genus zero Whitham hierarchy. My main previous result is the extension of this correspondence to the curves of higher genus.

Previous Results

Consider a large class of finite-dimensional solutions to the Whitham hierarchy called algebraic orbits. These solutions can be constructed as follows. One starts with a finite dimensional moduli space, called the universal configuration space, that consists of a curve Γ of genus g, punctures P_α, and a pair of Abelian integrals E and Q. Then one defines special coordinates on this space (Whitham times), picks up a leaf defined by fixing some of them, and maps this leaf to the moduli space of algebro-geometric data in such a way that coordinate lines go to Whitham flows. All the information about an algebraic orbit can be encoded in a single function $\tau(t)$ depending only on the moduli. This function is called the τ-function of an algebraic orbit. In the genus zero case Abelian integrals become polynomials, and if we choose an algebraic orbit with $Q = p$, then E can be identified (up to normalization) with the superpotential W, Ω_i define a basis of the corresponding Frobenius algebra, Whitham times t_i give flat coordinates on the orbit, and $F(t) = \log \tau(t)$ is a WDVV potential. This approach can be generalized to moduli spaces of curves of higher genus. One new feature of a higher genus case is a more complicated topology of the moduli space. As a result, the hierarchy has to be extended to include certain (multivalued) differentials $d\Omega_A$ that generate additional Whitham flows. Another important aspect is the choice of the normalization. Namely, in the genus zero case differentials $d\Omega_i$ are completely determined by their expansions in the neighborhoods of marked points P_α. In the higher genus case these conditions define $d\Omega_i$ only up to a holomorphic differential. This ambiguity is fixed by introducing a normalization condition. There are two main choices — real normalization, which is defined by $\Im\left[\oint_c d\Omega_i\right] = 0 \ \forall c \in H_1(\Gamma_g, \mathbb{Z})$ and complex normalization (or normalization w.r.t. a-cycles). Complex normalization requires making a choice of a canonical basis $\mathcal{B}$ of cycles in the homology of Γ_g and is then defined by $\oint_{a_k} d\Omega_i = 0 \ \forall a_k \in \mathcal{B}$. The Whitham equations were originally derived in [Kri88] for real-normalized differentials. However, after the relationship between Whitham equations and WDVV equations was found in [Kri92] and [Dub92], the focus shifted to the complex normalization condition [Kri94]. There were two main reasons for this. First, in the complex-normalized case the τ-function is holomorphic, which is important for string theory applications. Second, the derivation of the expression for the τ-function relied on the Riemann bilinear identities. Corresponding identities for real normalized differentials are technically more complicated. However, complex normalization has certain disadvantages. In particular, it is well-defined only on the extended moduli space that incorporates the choice of a canonical basis $\mathcal{B}$ into moduli data. In [Dzh00] I develop the real-normalization version of the above approach for a single puncture case. A corresponding algebraic orbit is defined by picking up a real leaf in the universal configuration space, introducing Whitham coordinates on this leaf and then mapping it into the moduli space in such a way that the resulting differentials are real-normalized. Then I prove the real-normalized version of the Riemann bilinear identities (generalized to multivalued differentials). Using these identities I find the explicit formula for the τ-function of an algebraic orbit and prove that its logarithm $F(t) = \log \tau(t)$. gives a solution to the WDVV equations and so defines a Frobenius manifold.

Current research

τ-function for Analytic Curves My main research interest at present is yet another unexpected link between the theory of integrable hierarchies and some very classical problems in *complex analysis* and *potential theory*. This link was recently discovered in a series of papers by P. Wiegnamm, A. Zabrodin, and their collaborators ([WZ00], [Zab01], [KKMW$^+$01]) and it has its origin in the *Laplacian growth problem* describing the motion of the interface front between two incompressible fluids of different viscosities, like oil and water. If the fluids are confined to the *Hele-Shaw cell* (see, for example, [Pel88]), the problem becomes two-dimensional and can be studied by means of *complex analysis*. Mathematically, it can be described as follows.

Consider a simply-connected domain $D \subset \mathbb{C}$ bounded by a simple closed curve γ and denote by $D^c \subset \bar{\mathbb{C}}$ its complement (on the Riemann sphere). We assume that $0 \in D$ and $\infty \in D^c$. The harmonic moments of the domain are then defined by

$$t_0 = \frac{1}{\pi}\text{Area}(D) = \frac{1}{\pi}\iint_D 1\, dA, \quad t_k = \frac{-1}{\pi}\iint_{D^c} \frac{z^{-k}}{k}\, dA, \qquad v_k = \frac{1}{\pi}\iint_D z^k\, dA,$$

where the moments t_k are called *exterior* and the moments v_k are called *interior*. If both sets of moments are known, the domain can be easily reconstructed (e.g., using Green's theorem we can easily see that t_k and v_k are just coefficients in the Laurent series expansion of the Schwarz function $S(z)$ of the curve γ). But in fact either set of moments can be used to (locally) characterize the domain. Let us assume that we know the harmonic moments of the exterior $\mathbf{t} = \{t_k\}_{k=0}^{\infty}$ and consider the following three closely related problems:

- *Richardson's Moment Problem* (or, equivalently, as *The Inverse Problem of 2D Potential Theory*): Reconstruct D (or D^c) from its harmonic moments. Alternatively, determine the interior moments $v_k = v_k(\mathbf{t})$.
- *Riemann's Mapping Problem*: Find a conformal map $w = w(z)$ of the exterior domain D^c in the z-plane to the exterior of the unit disk in the w-plane (such a map exists by *Riemann's Theorem*)
- *Dirichlet Boundary Value Problem*: Find Green's function $G(z_1, z_2)$ of the Laplace operator in D^c.

It turns out that all these questions can be explicitly answered in terms of a single function $F(\mathbf{t})$ as follows. Let us first introduce the $\mathcal{D}$-operator:

$$\mathcal{D}(z) = \sum_{k\geq 1} \frac{z^{-k}}{k} \partial_{t_k}, \quad \bar{\mathcal{D}}(\bar{z}) = \sum_{k\geq 1} \frac{\bar{z}^{-k}}{k} \partial_{\bar{t}_k}, \quad \mathcal{D}(z,\bar{z}) = \partial_{t_0} + \mathcal{D}(z) + \bar{\mathcal{D}}(\bar{z}).$$

Then we have:

- $v_k(\mathbf{t}) = \frac{\partial F}{\partial t_k}(\mathbf{t})$.
- $w(z;\mathbf{t}) = z \exp\left(-\frac{1}{2}\partial_{t_0}^2 F(\mathbf{t}) - \partial_{t_0}\mathcal{D}(z)F(\mathbf{t})\right)$.
- $G(z_1, z_2; \mathbf{t}) = \log|z_1^{-1} - z_2^{-1}| + \frac{1}{2}\mathcal{D}(z_1,\bar{z}_1)\mathcal{D}(z_2,\bar{z}_2)F(\mathbf{t})$.

The very surprising fact is that this function $F(\mathbf{t})$ satisfies the Hirota equations of the dispersionless Toda hierarchy, and therefore can be identified with the logarithm of the τ-function of this hierarchy. This correspondence uncovers the hidden integrable structure of the problem and identifies the dispersionless limit of the first Lax operator L of the $2D$ Toda hierarchy with $z(w)$, and the dispersionless limit of the second Lax operator $\bar{L}$ with $S(z(w))$. In particular, we can see that the evolution of the conformal map $z(w)$ w.r.t. t_k is Hamiltonian. The dispersionless Toda hierarchy is just a particular example of a Whitham hierarchy for a genus zero curve with two punctures. In this formalism the moments t_k are just Whitham times of the hierarchy. For multiply-connected domains we have to consider curves of higher genus. It is important to note that the normalization of the corresponding Whitham differentials has to be the same one as the one considered in my previous work. I am particularly interested in the following questions:

- Construction of explicit examples. Such examples allow one to compute the τ-function explicitly for some variables. In particular it is interesting to construct examples of ring-type domains corresponding to the elliptic curves.
- Study of the behavior of the τ-function when the boundary curves develops a singularity. In particular it is interesting to see if there are solutions that model the observable phenomena, like Saffman-Taylor fingering and pattern formation.
- Relationship with the classical Toda hierarchy.

Isomonodromy equations Another project that I am currently working on is the extension of the theory of isomonodromy equations to the higher genus curves. For genus zero case the study of such equations began in the works of R. Fuchs [Fuc07], P. Painlevé [Pai76], R. Garnier [Gar26], L. Schlesinger[Sch12], and others at the beginning of the XIX century, and more recently by M. Jimba, T. Miwa, and K. Ueno [JMU81]. Let A be a flat meromorphic connection on a rank r vector bundle V on a curve Γ. For simplicity assume that A is *logarithmic*, i.e., that all its singularities are simple poles $x_1, \ldots, x_n$. Let $\mathbf{e}$ be a frame for V at some chosen basepoint b_0. The choice of A allows us to parallel-translate $\mathbf{e}$ along any loop $\gamma \in \Gamma, \{x_1, \ldots, x_n\}$. The new frame $\mathbf{e}_\gamma$ obtained in such way differs from $\mathbf{e}$ by a multiplication by a non-degenerate matrix $C_\gamma \in \mathrm{GL}(r,\mathbb{C}) = \mathrm{Aut}(V)$. In this way we get the monodromy map that puts in correspondence to a connection A its monodromy data $\mathcal{M}_A : \pi_1(\Gamma - \{x_1, \ldots, x_n\}, b_0) \to \mathrm{GL}(r,\mathbb{C}) = Aut(V)$ (modulo conjugation). For connections that are not logarithmic monodromy data has to be modified to include the so-called Stokes matrices. *Isomonodromic deformation* is a deformation of a connection A that preserves its monodromy data $\mathcal{M}_A$. Isomonodromic equations are very interesting and important. Special classes of such equations are Schlesinger and Painlevé equations, whose relation with soliton theory is well-known, [IN86]. Also, in [Dub96b] B. Dubrovin used monodromy data to to classify a large class of Frobenius manifolds. Isomonodromic equations were recently considered from the point of view of the Whitham theory in [Tak98]. In addition, recently an interpretation of the isomonodromic equations as classical limits of the *Knizhnik–Zamolodchikov–Bernard equations* of *Conformal Field Theory* was considered in [LO99]. Unfortunately, most of the previous results are obtained for rational curves. Recently a new approach to these equations was suggested in [Kri]. It is based on the notion of *Tyurin parameters*. For a non-trivial vector bundle V on a curve Γ one can not take a basis of some fiber V_{b_0} and extend it to the whole Γ — there will be a number of points where this basis will degenerate. Tyurin parameters [Tju67] are parameters describing these degenerations. It turns out that all important objects that we are interested in — the bundle itself, the connection A, etc., can be explicitly described in terms of Tyurin parameters. In particular, it is possible to give a *Lax representation* of the isomonodromic deformations. Such an approach should be compared to a more traditional approach to these classes of problems, suggested by Hitchin [Hit87]. Following Hitchin, one starts with a *free Hamiltonian theory*, reduces it by some *group action*, and then constructs the Lax representation. We are going in the opposite direction — the Lax representation is the starting point, and the Hamiltonian aspects of the theory are then obtained with the help of a very general Hamiltonian theory of soliton equations constructed with the help of Baker-Akhiezer functions [KP98],[KP97]. This approach was recently successfully used by Krichever [Kri01] to construct an explicit parameterization of Hitchin systems in terms of Tyurin parameters. Note that isomonodromic deformations can be considered as the Whitham deformations of Hitchin systems. By such change of a viewpoint I hope to avoid certain difficulties of the traditional approach and get a better insight of what happens in the genus $g > 0$ case. I am mostly interested in constructing concrete examples. In particular, it will be very interesting to see what is a correct version of the famous *Painlevé equations* in the higher genus case.

References

[BK00] H. W. Braden and I. M. Krichever, editors. *Integrability: the Seiberg-Witten and Whitham equations.* Gordon and Breach Science Publishers, Amsterdam, 2000. Papers from the meeting held in Edinburgh, 1998.

[dB72] J. de Boussinesq. Théorie des ondes et des remours qui se propagent le long d'un canal rectangulaire horizontal, et communiquant au liquide contenu dans ce canal des vitesses sensiblemant parielles de la surface au fond. *J. Math. Pures et Appliqueés*, 17(2):55–108, 1872.

[DN74] B. A. Dubrovin and S. P. Novikov. Periodic and conditionally periodic analogs of the many-soliton solutions of the Korteweg-de Vries equation. *Ž. Èksper. Teoret. Fiz.*, 67(6):2131–2144, 1974.

[Dub92] B. A. Dubrovin. Hamiltonian formalism of Whitham-type hierarchies and topological Landau-Ginsburg models. *Comm. Math. Phys.*, 145(1):195–207, 1992.

[Dub96a] Boris Dubrovin. Geometry of 2d topological field theories. In *Integrable systems and quantum groups (Montecatini Terme, 1993)*, volume 1620 of *Lecture Notes in Math.*, pages 120–348. Springer, Berlin, 1996.

[Dub96b] Boris Dubrovin. Painlevé transcendents and two–dimensional topological field theory. In *The Painlevé property: one century later*, Proceedings of 1996 Cargèse school (to appear), 1996.

[DVV91a] Dijkgraaf, Robbert, Verlinde, Herman, and Verlinde, Erik. Notes on topological string theory and 2d quantum gravity. In *String theory and quantum gravity (Trieste, 1990)*. World Sci. Publishing, River Edge, NJ, 1991.

[DVV91b] Dijkgraaf, Robbert, Verlinde, Herman, and Verlinde, Erik. Topological strings in $d < 1$. *Nuclear Phys. B*, 352(1):59–86, 1991.

[Dzh00] Anton Dzhamay. Real-normalized Whitham hierarchies and the WDVV equations. *Internat. Math. Res. Notices*, (21):1103–1130, 2000.

[FPU74] E. Fermi, J. Pasta, and S. Ulam. Studies of nonlinear problems I, Los Alamos Sci. Lab. Rep. LA-1940(1955). In Alan C. Newell, editor, *Nonlinear wave motion*, volume 15 of *Lectures in Applied Mathematics*, pages 143–156, Providence, R.I., 1974. American Mathematical Society.

[Fuc07] R. Fuchs. Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegenen wesentlich singulären Stellen. *Math. Annalen*, 63:301–323, 1907.

[Gar26] R. Garnier. Sur des équations différentielles du troisième order dont l'intégral générale et uniforme et sur une class d'équations nouvelles dódre supérieur dont l'intégrale générale a ses points critiques fixes. *Ann. Sci. École Norm. Sup.*, 43(3):177–307, 1926.

[GGKM67] Clifford S. Gardner, John M. Greene, Martin D. Kruskal, and Robert M. Miura. Method for solving the Korteweg-deVries equation. *Phys. Rev. Lett.*, 19:1095–1097, 1967.

[Hit87] Nigel Hitchin. Stable bundles and integrable systems. *Duke Math. J.*, 54(1):91–114, 1987.

[IM75] A. R. Its and V. B. Matveev. Schrödinger operators with the finite-band spectrum and the N-soliton solutions of the Korteweg-de Vries equation. *Teoret. Mat. Fiz.*, 23(1):51–68, 1975.

[IN86] Alexander R. Its and Victor Yu. Novokshenov. *The isomonodromic deformation method in the theory of Painlevé equations*, volume 1191 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.

[JMU81] Michio Jimbo, Tetsuji Miwa, and Kimio Ueno. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. General theory and τ-function. *Phys. D*, 2(2):306–352, 1981.

[KdV95] D. J. Kortweg and G. de Vries. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Philos. Mag. Ser.*, 39(5):422–443, 1895.

[KKMW+01] I. K. Kostov, I. Krichever, M. Mineev-Weinstein, P. B. Wiegmann, and A. Zabrodin. The τ-function for analytic curves. In *Random matrix models and their applications*, volume 40 of *Math. Sci. Res. Inst. Publ.*, pages 285–299. Cambridge Univ. Press, Cambridge, 2001. hep-th/0005259.

[Kon92] Maxim Kontsevich. Intersection theory on the moduli space of curves and the matrix Airy function. *Comm. Math. Phys.*, 147(1):1–23, 1992.

[KP97] Krichever, I. M. and Phong, D. H. On the integrable geometry of soliton equations and $n = 2$ supersymmetric gauge theories. *J. Differential Geom.*, 45(2):349–389, 1997.

[KP98] Krichever, I. M. and Phong, D. H. Symplectic forms in the theory of solitons. In *Surveys in Differential Geometry*, volume IV. International press, 1998.

[Kri] I. Krichever. Isomonodromy equations on algebraic curves. (private communications).

[Kri77] I. M. Krichever. Methods of algebraic geometry in the theory of nonlinear equations. *Uspehi Mat. Nauk*, 32(6(198)):183–208, 287, 1977.

[Kri88] I. M. Krichever. The averaging method for two-dimensional "integrable" equations. *Functional Anal. Appl.*, 22(3):200–213, 1988.

[Kri92] I. Krichever. The dispersionless Lax equations and topological minimal models. *Comm. Math. Phys.*, 143(2):415–429, 1992.

[Kri94] I. M. Krichever. The τ-function of the universal Whitham hierarchy, matrix models and topological field theories. *Comm. Pure Appl. Math.*, 47(4):437–475, 1994.

[Kri01] I. Krichever. Vector bundles and Lax equations on algebraic curves. hep-th/0108110v1, August 2001.

[Lax68] Peter D. Lax. Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.*, 21:467–490, 1968.

[LO99] A. M. Levin and M. A. Olshanetsky. Hierarchies of isomonodromic deformations and Hitchin systems. In *Moscow Seminar in Mathematical Physics*, pages 223–262. Amer. Math. Soc., Providence, RI, 1999.

[MvM75] H. P. McKean and P. van Moerbeke. The spectrum of Hill's equation. *Invent. Math.*, 30(3):217–274, 1975.

[Nov74] S. P. Novikov. A periodic problem for the Korteweg-de Vries equation. I. *Funkcional. Anal. i Priložen.*, 8(3):54–66, 1974.

[Pai76] P. Painlevé. *Œvres t. I, II, III.* SNRS, Paris, 1976.

[Pel88] Pierre Pelcé, editor. *Dynamics of curved fronts.* Perspectives in Physics. Academic Press Inc., Boston, MA, 1988.

[Rus44] J. Scott Russell. Report on waves. In *Rept. Fourteenth Meeting of the British Assocoation for the Advancement of Science*, pages 311–390, London, 1844. John Murray.

[Sch12] L. Schlesinger. Über eine Klasse von Differentialsystemen beliebliger Ordnung mit festen kritischen Punkten. *J. für Math.*, 141:96–145, 1912.

[Shi86] Takahiro Shiota. Characterization of Jacobian varieties in terms of soliton equations. *Invent. Math.*, 83(2):333–382, 1986.

[Tak98] Kanehisa Takasaki. Spectral curves and Whitham equations in isomonodromic problems of Schlesinger type. *Asian J. Math.*, 2(4):1049–1078, 1998. Mikio Sato: a great Japanese mathematician of the twentieth century.

[Tju67] A. N. Tjurin. The classification of vector bundles over an algebraic curve of arbitrary genus. *Amer. Math. Soc. Translat*, 63:245–279, 1967.

[TT95] Kanehisa Takasaki and Takashi Takebe. Integrable hierarchies and dispersionless limit. *Rev. Math. Phys.*, 7(5):743–808, 1995.

[Wit90] Edward Witten. On the structure of the topological phase of two-dimensional gravity. *Nuclear Phys. B*, 340(2-3):281–332, 1990.

[Wit91] Edward Witten. Two-dimensional gravity and intersection theory on moduli space. In *Surveys in differential geometry (Cambridge, MA, 1990)*, pages 243–310. Lehigh Univ., Bethlehem, PA, 1991.

[WZ00] P. B. Wiegmann and A. Zabrodin. Conformal maps and integrable hierarchies. *Comm. Math. Phys.*, 213(3):523–538, 2000. hep-th/9909147.

[Zab01] A. V. Zabrodin. The dispersionless limit of the Hirota equations in some problems of complex analysis. *Teoret. Mat. Fiz.*, 129(2):239–257, 2001. math.CV/0104169.