Normal Probability Distributions

The goal of this activity is to understand the properties of normal distributions.

Let the random variable $X$ denote the ideal symmetric distribution with mean $\mu$ and standard deviation $\sigma$. This distribution is known as the normal distribution. Statisticians have found that the probability density function of such a distribution is given by the function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

The most fundamental probability density function of this type has a mean of 0 and a standard deviation of 1. This is called the standard normal distribution and is given by the function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

1. Enter this function in your graphing calculator as Y1 and graph it in the viewing widow [-4.7,4.7] by [-0.5,0.5]

   \[ Y_1 = \frac{1}{\sigma} \sqrt{2\pi} e^{(x)} \]

2. Sketch the graph of the function.

3. Use the \texttt{\textasciitilde f(x)dx} command under the CALC menu to calculate the area under the curve from -1 to 1. How does this compare to the Empirical Rule?

4. Use the \texttt{\textasciitilde f(x)dx} command under the CALC menu to calculate the area under the curve from 1 to 2. How does this compare to what you would expect with the Empirical Rule?
5. In $Y_2$, enter the normal distribution with mean 0 and standard deviation 2 and graph it in the same widow with $Y_1$. Describe the similarities and differences of the two graphs.

$$Y_2 = \frac{1}{\sqrt{2\pi}} e^{\left(\frac{-x^2}{8}\right)}$$

6. Calculate the area under the curve of $Y_2$ from -2 to 2. Compare and contrast this answer to the one in item 3 above?

7. In $Y_3$, enter the normal distribution with mean 3 and standard deviation 1 and graph it in the same widow with $Y_1$. Describe the similarities and differences of the two graphs.

$$Y_3 = \frac{1}{\sqrt{2\pi}} e^{\left(\frac{(x-3)^2}{2}\right)}$$

8. In $Y_4$, enter the derivative of the standard normal distribution. The syntax is given below. Graph the derivative with the parent function (the normal curve) in the same widow.

$$Y_4 = \text{nDeriv}(Y_1, X, X)$$

a. Identify any $x$-intercepts of the derivative function. How are these points interpreted on the normal curve?

b. Identify any maximum or minimum values of the derivative function. How are these points interpreted on the normal curve?

c. Describe the behavior of the normal curve as $x$ increases, without bound. Describe the behavior of the normal curve as $x$ decreases, without bound.
Normal Probability Distributions

Properties of the Normal Probability Density Function

1. It is symmetric about its mean, $\mu$.
2. Because the mean = median = mode, the maximum point occurs at $x = \mu$.
3. It has inflection points at $\mu - \sigma$ and $\mu + \sigma$.
4. The area under the curve is 1.
5. The area under the curve to the right of $\mu$ equals the area under the curve to the left of $\mu$, which is $\frac{1}{2}$.
6. As $x$ increases, without bound, the graph approaches, but never reaches, the horizontal axis. As $x$ decrease without bound, the graph approaches, but never reaches, the horizontal axis.
7. The Empirical Rule: Approximately 68% of the area under the normal curve is between $x = \mu - \sigma$ and $x = \mu + \sigma$. Approximately 95% of the area under the curve is between $x = \mu - 2\sigma$ and $x = \mu + 2\sigma$. Approximately 99.7% of the area under the curve is between $x = \mu - 3\sigma$ and $x = \mu + 3\sigma$.

Interpretation of the area under the curve of a normal probability density function is given below.

The Area under a Normal Curve

Suppose a random variable $X$ is normally distributed with mean $\mu$ and standard deviation $\sigma$. The area under the normal curve for any interval of values of the random variable $X$ represents either

- the proportion of the population with the characteristics describes by the interval of values, or
- the probability that a randomly selected individual from the population will have the characteristics described by the interval of values.

9. The lengths of human pregnancy are normally distributed with $\mu = 266$ days and $\sigma = 16$ days.

a. Appropriately label the graph on the right.

b. Shade the region that represents the proportion of lengths of human pregnancy lasting longer than 290 days.

c. Suppose the area under the curve to the right of $X = 290$ days is 0.0668. Provide two interpretations of this result.