Infinitesimals:
Intuition and Rigor

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The infinitesimal has played an interesting role in the history of analysis. It was initially used to support the work of Newton and Leibniz in the development of the calculus. However, by the end of the 1700s, it became an object of derision and was finally driven away by the development of the concept of a limit and the epsilon-delta definitions that are still used today. But in 1960, a man named Abraham Robinson resurrected the infinitesimal and in the process heralded a new era in analysis, nonstandard analysis (NSA).

Isaac Newton (1643-1727) ended up using three different methods to justify his calculus 1) infinitesimals 2) fluxions and 3) the method of prime and ultimate ratios (Cultural). He defined a fluxion as the speed with which a quantity changes over time and denoted the fluxion of a variable *x* with $\dot{x}$. He used *o* to represent an infinitely small amount of time and specified that in this infinitely small time, *o*, a variable *x* will become $x+\dot{x}o$. In an example that Newton gives for a procedure that today we would say defines the differential equation that satisfies the curve of a given equation, we can see how Newton handled the infinitesimal *o* when working with the quantity $x+\dot{x}o$. In this example, he ends up with a long equation in which some of the terms contain *o* as a factor: $3x^2\dot{x}+3\dot{x}x^2o+\dot{x}^3o-2ax\dot{x}-a\dot{x}^2o+ay\dot{x}+ax\dot{y}+a\dot{x}\dot{y}o-3y^2\dot{y}-3\dot{y}y^2o-\dot{y}^3o^2=0$(Boyer, 1991, p. 397). As the final step in deriving his results he states, "but further, since *o* is supposed to be infinitely small so that it [will] be able to express moments of quantities, terms which have it as a factor will be equivalent to nothing in respect of the others. I therefore cast them out." The resulting equation is $3x^2\dot{x}-2ax\dot{x}+ay\dot{x}+ax\dot{y}-3y^2\dot{y}=0$. This "casting out" is not justified in terms of a limit argument, but only in an intuitive sense of how they should behave (Katz, 1993, p. 467).

In his third published account of his calculus, Newton sought to avoid infinitesimals, although he retained the notation *o*, which still retained its magical disappearing property. Instead of infinitesimals, Newton used the doctrine of "prime and ultimate ratios" and proved the power property for finding a derivative (Cultural). He started by specifying the ratio of *o* to the difference of x^n and x^n after the infinitely small time *o* and then divided this ratio by $\frac{o}{o}$ and ended up with the result as shown:

$$\frac{o}{(x+o)^n - x^n} \div \frac{o}{o} = \frac{1}{x^n - x^n + nx^{n-1} + \frac{n(n-1)}{2} ox^{n-2} + \cdots} = \frac{1}{nx^{n-1}}$$

To find the prime and ultimate ratio, Newton lets *o* vanish, obtaining the ratio $1:(nx^{n-1})$. This use of *o* more closely resembles the idea of a limit. As Newton writes in *Principia*, "Those ultimate ratios with which quantities vanish are not truly the ratios of ultimate quantities…but limits towards which the ratios of quantities decreasing without limit do always converge, and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished *in infinitum*." But mathematicians for the next century were bothered by the thought that a ratio could exist between increments that vanish. (Boyer, 1991, p. 397)

Newton is considered to have developed his calculus ten years prior to Gottfried Leibniz (1646-1716), but Leibniz was the first to publish his calculus. In this first publication entitled *A New Method for Maxima and Minima, and also for Tangents, which is not Obstructed by Irrational Quantities*, (1684) Leibniz presents the formulas for the product rule, quotient rule, and power rule for finding derivatives. He also presents notation that is largely used today. In lieu of Newton's notation of $\dot{x}o$ to represent a small change in *x*, Leibniz uses the familiar *dx*. His way of handling the idea that very small quantities vanish is to "cast out" terms that have more than one infinitesimal as a factor. For example, he derives the product rule by demonstrating that the smallest difference in *xy* is represented by *dxy*, which is equal to $(x+dx)(y+dy)-xy$. Leibniz argues that since *dx* and *dy* are infinitesimally small, their product must be infinitely infinitely small, and can therefore be disregarded (Boyer, 1991, p. 404). Thus, $(x+dx)(y+dy)-xy = xy+ydx+xdy+dxdy-xy = xdy+ydx$. In Leibniz's work, the infinitesimal *dx* has four slightly different interpretations 1) *dx* is indistinguishable from zero 2) *dx* is neither equal to nor not equal to zero 3) dx^2 is equal to zero and 4) *dx* becomes vanishingly small (Bell, 1985). In addition, Leibniz was bothered by the fact that his infinitesimals did not obey the Archimedean property.

A Dutch geometer and physician Bernard Nieuwentijt (1654-1718) published three treatises from 1694-1696 criticizing the vagueness of Newton's evanescent quantities and offering his own account of infinitesimals. His account treats infinitesimals as constants, not variables. In particular, he postulates adding to the domain of finite numbers infinitesimal and infinite quantities, with infinitesimal quantities being those that are less than any given finite quantity and infinite quantities being those that are greater than any given finite quantity. This entire domain is subject to an ordering relation of greater than and less than. All quantities except zero satisfy the Archimedean principle that a quantity can be multiplied sufficiently many times to be greater than any given quantity (Vermeulen, 1985). Nieuwentijt's version of infinitesimals is strikingly similar to the 20th century version of infinitesimals.

Nieuwentijt's objections to Newton's infinitesimals appears to have received far more notice than his account of infinitesimals. In general, Nieuwentijt felt that Newton's view of infinitesimals was dangerous because it led to human infringement on God's domain and worse, to atheism. Nieuwentijt argued that God created humans so that "the human intellect is not capable of rising to a true and adequate understanding of the infinite itself", so that humans should not dwell in the infinite realm. He also rejected higher-order infinitesimals as they presupposed that matter can be infinitely divided leading to the conclusion that matter is indestructible, but this was tantamount to atheism (Vermeulen, 1985).

The most well-known person to object to Newton's and Leibniz's use of infintesimals was the Irish philosopher Bishop George Berkeley (1685-1753). Berkeley published his critique of the theory underlying the calculus in *The Analyst* in 1734. He addressed it to "an infidel mathematician", who is presumed to be Edmund Halley. Berkeley considered Halley an infidel for persuading a friend that the doctrines of Christianity were inconceivable. Berkeley quoted Newton who said, "the minutest errors are not to be neglected in mathematics" and then questioned how certain quantities can be said to vanish. He directly addressed Newton's derivation of the prime ratio

$$1:\left[nx^{n-1}+\frac{n(n-1)}{2}ox^{n-2}+\cdots\right]$$

He critiques the vanishing terms by stating

> But it should seem that this reasoning is not fair or conclusive. For when it is said, let the increments vanish, i.e., let the increments be nothing, or let there be no increments, the former supposition that the increments were something, or that there were increments, is destroyed, and yet a consequence of that supposition, i.e., an expression got by virtue thereof, is retained.

Berkeley claimed he simply could not fathom an infinitely small quantity. "But to conceive a part of such infinitely small quantity that shall be still infinitely less than it, and consequently though multiplied infinitely shall never equal the minutest finite quantity, is, I suspect, an finite difficulty to any man whatsoever" (Katz, 1993, p. 526).

There is some evidence that Berkeley had read the work of Nieuwentijt and many of the same religious themes run through the work of both men (Vermeulen, 1985).

The most important response to Berkeley's criticisms came from Colin Maclaurin (1698-1746) in his *Treatise of Fluxions* (1742). Maclaurin put forth very strong arguments in defense of the infinitesimal. He defended the notion that a finite quantity can vanish by noting that the limit of a ratio can be determined as the increments are decreased. He even defined the tangent as a limit: "The tangent…is the…line that limits the position of all the secants that can pass through the point of contact, though strictly speaking it be no secant, [just as] a ratio may limit the variable ratios of the increments, though it cannot be said to be the ratio of any real increments." He grounded his arguments in the "manner of the ancients" using geometry, including the method by exhaustion and its often associated *reductio ad absurdum*, as used by Archimedes. His defense totaled 754 pages and because of its length and "ancient" approach was largely ignored by eighteenth century mathematicians (Katz, 1993, p.529).

Even the great Euler (Leonhard Euler, 1707-1783) believed that the differentials could be both equal to zero and qualitatively different than zero. Euler published a two volume work called *Introduction to Analysis of the Infinite* in 1748. He felt that any finite ratio could be equal to the ratio 0:0 and that "the calculus of the infinitely small is …nothing but the investigation of the geometric ratio between different infinitely small quantities" (Katz, 1993, p. 529). Jean Le Rond d'Alembert (1717-1783) disagreed with Euler's view and considered limits to be at the heart of calculus. He defined limit in an article he wrote for the *Encyclopédie* in 1754 as: "One magnitude is said to be the limit of another magnitude when the second may approach the first within any given magnitude, however small, though the first magnitude may never exceed the magnitude it approaches" (Katz, 1993, p. 530). Like Maclaurin's before him, though, d'Alembert's ideas were disregarded by eighteenth century mathematicians and the use of infinitesimals, fluxions, and the ratio of zeroes continued to the justifications for the workability of the calculus.

Near the end of the eighteenth century, two more mathematicians attempted to find a more rigorous foundation for analysis other than the infinitesimal. Joseph Lagrange (1736-1813) not only attempted to remove infinitesimals from the definition of a derivative, he rejected the notion of a limit as well. He published his ideas in 1797 in a text with a title to fully explicate his goal: *The Theory of Analytic Functions, containing the principles of the differential calculus, released from every consideration of the infinitely small or the evanescent, of limits or of fluxions, and reduced to the algebraic analysis of finite quantities*. Lagrange asserted that any function could be written as a power series. So given $y = f(x)$ to be any function and i to be an indeterminate value, then

$$f(x+i) = f(x) + pi + qi^2 + ri^3 + \cdots$$

where p, q, and r are functions of x. He then showed that the differential $\frac{dy}{dx}$ was the same as the function $p(x)$ and created the familiar notation $f'(x)$ to represent the derivative. He even went on to show that $f'' = q$ and $f''' = r$, etc. However, as is obvious to the modern reader, Lagrange's theories were flawed by the fact that not all functions can be represented as a power series, though his peers objected to his work primarily because of the lengthy calculations that were often involved. In addition, Lagrange did note that since the power series explained the differential that the use of the earlier differential techniques was still valid. Thus, his techniques were not often used, even by himself (Katz, 1993, p. 530).

The other mathematician who published in 1797 was Lazare Carnot (1753-1823). His book *Réflexions sur la métaphysique du calcul infinitesimal* became extremely popular and was published in several languages. His attempt to put aside the “fluxions of Newton, the differentials of Leibniz, and the limits of d’Alembert” was to use “the principles of the compensation of errors” (Katz, 1993, p. 481). He felt that infinitesimals were to be used like imaginary numbers and introduced as needed and eliminated when a final result is reached. In particular, he believed that infinitesimals of higher order could be eliminated and that this simplification was quite similar to the ancient method of exhaustion. In response to the objection that infinitesimals could not be both zero and non-zero he gave the rather vague explanation that “what are called infinitely small quantities are not simply any null quantities at all, but rather null quantities assigned by a law of continuity which determines the relationship” (Katz, 1993, p. 481).

As the nineteenth century dawned, two forces pushed towards the creation of more rigor in analysis. One was the notion seen in Lagrange’s work that analysis should not have its foundations either in science or in other branches of mathematics, but solely in the arithmetic of the natural numbers. The other force was pedagogy. As some of the great mathematicians set out to teach analysis, they became bothered by the lack of quality deductive proofs to support the theories they were teaching. This latter force was a motivation for contributions to analysis by Cauchy, Weierstrass, and Dedekind (Grabiner, 1981, p. 47).

Augustin-Louis Cauchy (1789-1857) published the textbook *Cours d’analyse* in 1821, which is now considered the foundational textbook of modern analysis. The textbook was derived from lectures he gave at École Polytechnique, a leading school for mathematicians in France. Though many of Cauchy’s ideas were derived from the works of other mathematicians, Cauchy was able to pull together two key facts. The first was that a limit could be understood using algebraic inequalities (the ε, δ definition we know today) and the second was that all of the calculus could be explained in terms of limits. (Grabiner, 1981, p. 77)

The definition that Cauchy gave for a limit was "when the values successively attributed to the same variable indefinitely approach a fixed value in such a way as to end by differing from it as little as one wishes, this latter is called the limit of all the others (Bottazzini, 1986, p. 103). From this definition, he defined infinitesimal as "when the successive numerical values of the same variable decrease indefinitely in such a way as to fall below any given number, this variable becomes what one calls an infinitesimal or an infinitely small quantity. A variable of this kind has zero as a limit" (Bottazzini, p. 103).

And so the definition of an infinitesimal was finally put on a rigorous algebraic foundation which is still found in the analysis taught today. Thus might have ended the story of the infinitesimal except for the sudden insight of Abraham Robinson (1918-1974) as he walked across Princeton, 140 years after the publication of Cauchy's *Cours d'Analyse*. Robinson had been working with nonstandard arithmetic and modeling and most likely this work inspired him to see that a rigorous foundation of the calculus can be made with infinitesimals. From this inspiration, nonstandard analysis was born (Dauben, 1995, p. 277).

In 1966, Robinson published *Nonstandard Analysis*, which had as its frontispiece the same image as the one on Euler's book about analysis published in 1748. In his book, Robinson states his purpose is to show how infinitesimals "appeal naturally to our intuition". He goes on to say "It is shown in this book that Leibniz' ideas can be fully vindicated and that they lead to a novel and fruitful approach to classical Analysis and to many other branches of mathematics" (Dauben, 1995, p. 346).

Nonstandard analysis starts by defining the structure of the hyperreal numbers by specifying two axioms:

Axiom A – **R** is a complete ordered field.

Axiom B – $\mathbf{R}^*$ is a proper ordered field extension of **R**.

$\mathbf{R}^*$ contains not only the field of real numbers (**R**), but also new numbers called infinitesimals and infinites. Infinitesimals are defined as an element of $\mathbf{R}^*$, *x*, if $|x| < r$ for all positive real *r*. Further, two elements $x, y \in \mathbf{R}^*$ are said to be infinitely close if $x \approx y$ and if $x - y$ is an infinitesimal. Thus, *x* is an infinitesimal if and only if $x \approx 0$. An element, *x*, of $\mathbf{R}^*$ is an infinite if $|x| > r$ for all real *r* (Keisler, 1976, p. 2).

One of the fundamental properties of nonstandard analysis is the transfer principle. This principle states that every proposition that is true in classical analysis is also true in nonstandard analysis. (Keisler, 1976, p. 31) Thus, all the ε, δ definitions in classical analysis can be described with nonstandard analysis. Here is a standard ε, δ definition of a limit: the limit of $f(x)$ as $x \to c$ is *L* if for every real $\varepsilon > 0$ there is a real $\delta > 0$ such that whenever *x* is real and $0 < |x - c| < \delta$, $|f(x) - L| < \varepsilon$. The equivalent definition in nonstandard analysis is: whenever $x \approx c$ but $x \neq c$ then $f(x) \approx L$ (Keisler, p. 103).

In retrospect, one can see the germ of nonstandard analysis in Niewentijt's account of infinitesimals. Robinson was a great study of the history of analysis and was familiar not only with the works of Newton and Leibniz, of course, but also with the works of Niewentijt and Berkeley. Thus, Robinson did address the issue of whether or not infinitesimals obeyed the Archimedean axiom in nonstandard analysis. However, the answer to this question is a little tricky. As Robinson once noted, "In our own theory the answer to the question whether the

Archimedes' axiom is true not only in **R** but also in $\mathbf{R}^*$ is unambiguously, yes – and no!" If the axiom is taken to be that every for every pair of numbers *a* and *b* such that *a* is less than *b* there is a natural number *n* for which $an > b$, then since this is true in **R**, it holds for $\mathbf{R}^*$ as well when *n* is an infinite natural number. If the axiom is taken to mean that that *a* can be added *n* times to be greater than *b*, then the postulate does not hold in $\mathbf{R}^*$ (Dauben, 1995, p. 351).

Nonstandard analysis was able to avoid Berkeley's objection that infinitesimals were the "ghost of departed quantities" by dealing with the equivalence of two quantities rather than their equality. Consequently, nonstandard analysis asserted that $dy/dx \cong f'(x)$ not that $dy/dx = f'(x)$. Equality was only asserted for the standard part of dy/dx and $f'(x)$ (Dauben, 1995, p. 351).

The power of nonstandard analysis has thus far not been in creating powerful new results, but rather in its ability to allow mathematician to pursue theorems in a simpler and often more intuitive manner than could be done in classical analysis. Although, practitioners of nonstandard analysis are hopeful it will yet yield profound contributions to mathematics (S. Leth, personal communication October, 10, 2006). It is interesting to see how analysis can be based on both a constructivist and intuitive view of the infinitesimal. While the future of NSA is uncertain, its creation helps to explain how the calculus was able to develop on what for a long time appeared to be the unstable edifice of the infinitesimal.

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