An Inquiry into the Influences of Pedagogical Content Knowledge and Epistemological Stance on Teachers’ Approach to Teaching Fundamental Theorem of Calculus

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Abstract
This qualitative study examined four university mathematics instructors with differing levels of educational and teaching experience. We considered data from our individual interviews with these four instructors. The analysis of these interviews leads us to discover that common influences on instructors’ approach to the FTC include KCS, SCK, CCK, and personal epistemological beliefs. However, the impact of these factors may differ from instructor to instructor. The data also suggests that PCK consists of overlapping components, contrary to Hill, Ball, & Schilling (2008) construct of PCK.

Introduction

Statement of the Problem
In mathematics education research, the ultimate goal of any study is to understand the nature of mathematical thinking, teaching, and learning so that we may use this knowledge to improve the teaching of mathematics (Schoenfeld, 2000). Teachers play a central role in the process of teaching and learning, as Speer and Wagner (2009) described, “Mathematics teachers have a responsibility to direct and shape the learning opportunities for their students” (p. 530). For this reason researchers focused a significant amount of literature to examine teachers’ knowledge (Hill, Ball, & Schilling, 2008; Hill, Sleep, Lewis, & Ball, 2007).

Teachers need to possess various kinds of knowledge. These kinds of knowledge include, but not limited to, knowledge of students’ challenges, such as conceptions and misconceptions, while learning mathematical contents. Shulman (1986) referred to this kind of knowledge as pedagogical content knowledge (PCK). Shulman argued that “If those preconceptions are misconceptions, which they so often are, teachers need knowledge of the strategies most likely to
be fruitful in reorganizing the understanding of learners, because those learners are unlikely to appear before them as blank slates” (p. 9).

Understanding the key ideas in mathematics is fundamental for learning mathematics (National Council of Teachers of Mathematics, 2000). If students understand these important concepts, it opens an opportunity for the students to build upon these foundations. In Calculus, the *Fundamental Theorem of Calculus* (FTC) is the corner stone to developing an understanding of the relationship between differentiation and integration. It gives the precise relationship between the derivative and the integral (Stewart, 2006). “The FTC deserves its high-sounding name for both theoretical and practical reasons. It’s fundamental theoretically because it connects two main concepts of the calculus: the derivative and the integral.

It is easy to lose sight of the broad and conceptual role of the FTC in an otherwise skills-oriented calculus course. Emphasis on skills is common since “skills are easy to test for, and tests for skills are easy to defend” (Schoenfeld, 2007, p. 72). However, skills-based problems often suffer from “absence of human meaning or purpose, which may inhibit both motivation and imagination” (Ruthven, 1994, p. 440), leaving “significant issues with regard to ... conceptual understanding” (Schoenfeld, 2007, p. 72). This is especially problematic since “students take tests as models of what they are to know,” so “assessment shapes what students attend to, and what they learn” (Schoenfeld, 2007, p. 72).

*Research Question*

This study mainly attempted to answer the following research question: How do a teacher’s epistemological stance and PCK inform an influence the presentation and content involved in their instruction of the FTC in undergraduate courses?
This kind of research will be valuable to teachers by developing self-awareness about their knowledge. The results of this study could be extended in the future through quantitative instruments to develop measures for teachers’ PCK.

Strategy of Inquiry

Individual members of our team conducted semi-structured one-on-one interviews (Patton, 2002). The interviews were divided into two parts: Background questions in which we asked the participants about their educational and professional experiences. This reason of this part is to help us locate the participant in relation to other participants. The second part was focused on mathematical knowledge for teaching in which we were looking for teachers’ perceptions of knowledge and how these perceptions inform the way they approach the FTC. Also, we included some task-based interview items trying to investigate teachers’ explanation of students’ common mistakes in order to develop some understanding of their knowledge of content and students.

Review of Literature

Teachers’ Knowledge

Even and Tirosh (2002) argued that teachers need to possess various kinds of knowledge. More than two decades ago, Shulman (1986) was interested in studying the knowledge and its growth in the minds of teachers. So, he distinguished among three categories of knowledge: subject matter content knowledge, PCK, and curricular knowledge. By the content knowledge, he referred to the amount and organization of knowledge in the mind of teacher. Shulman argued that this kind of knowledge requires going beyond knowledge of the facts or concepts of a domain which requires the ability to explain why a particular proposition is worth knowing and how it relates to other propositions both in theory and practice. So, we can conclude that teachers need to possess a deep understanding of the subject that they teach. Since, if they do not possess
this kind of knowledge, they will, less likely, be able to help their students. But is this kind of knowledge sufficient for teachers to offer this kind of help to their students?

One of the six principles in the NCTM (2000) standards for school mathematics is the curriculum principle. NCTM (2000) stated that “an effective mathematics curriculum focuses on important mathematics – mathematics that will prepare students for continued study and for solving problems in a variety of school, home, and work settings” (p.14). This was the second type of knowledge that Shulman defined as the *curricular knowledge*. By *curricular knowledge*, Shulman referred to knowledge of programs designed for teaching particular subjects and topics at a given level and the variety of instructional materials available in relations to those programs. So, the second major component requires that teachers should be provided by sufficient knowledge about curriculum and curricula plans.

*Pedagogical Content Knowledge*

In effective teaching, teachers need to draw upon different kinds of knowledge. This is what Peterson et al. (1989) went to in a study on teachers’ knowledge and beliefs. They argued that “As persons whose daily task is to understand and interpret the rapid flow of events in a classroom, and to make decisions and act on their interpretations, teachers obviously rely on their knowledge and beliefs” (p.3).

A third category of teachers’ knowledge in Shulman’s (1986) study was PCK. He introduced this concept to refer to teachers “understanding of what makes the learning of specific topic easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons” (p.9). He also argued that “If those preconceptions are misconceptions, which they so often are, teachers need knowledge of the strategies most likely to be fruitful in reorganizing the
understanding of learners, because those learners are unlikely to appear before them as blank slates (p. 9).

In another study on this kind of knowledge, Shulman (1987) defined PCK as “the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction” (Shulman, 1987, p.8). From this study, Shulman argued that a proper understanding of this kind of knowledge will make the emergence of pedagogically excellent teachers.

Due to its importance in mathematics teaching, PCK drew the researchers’ interest for the past two decades. Many scholars assumed its existence and contribution to effective teaching and student learning. Its importance appears in the literature devoted to study it.

Hill, Rowan, and Ball (2005) studied the effect of teachers’ mathematical knowledge for teaching on students’ achievement. The researchers found that teachers’ content knowledge for teaching was a significant predictor of student achievement. This finding suggested that this kind of knowledge plays a role even in the teaching of very elementary mathematics content. The researchers implied that “If successful, efforts to improve teachers’ mathematical knowledge through content-focused professional development and pre-service programs will improve student achievement, as intended” (p.400).

More recently, Hill, Ball, and Schilling (2008) tried to measure, conceptualize and improve PCK. They studied teachers’ knowledge of students and content (KCS) using results from multiple choice items. They reported many results regarding this kind of knowledge. Among these results, they reported that this domain remains under-conceptualized and understudied and further investigation is needed.
In a more recent study on the undergraduate math teaching, Speer and Wagner (2009) examined elements of knowledge for teaching needed by a mathematician to orchestrate whole-class discussions in an undergraduate classroom. They tried to examine the knowledge that teachers could employ to make use of students’ mathematical ideas to guide whole-class discussions in ways that further the mathematical goals for the class. This study tried to answer the following question: “How can teachers employ their PCK and specialized content knowledge (SCK) to follow students’ mathematical reasoning to further the learning goals of the class?”

Speer and Wagner (2009) found that the lack of PCK disabled the teacher from using students’ contributions as an opportunity for analytic scaffolding that would have helped the discussion progress toward the mathematical goal. Which means substantial knowledge of mathematical content is not all that is needed to turn contributions from students into the building blocks of a productive discussion which in turn contributes to the students’ achievement.

*What is Epistemology?*

Epistemology is the study of the nature of knowledge, its validity, and its transfer (Hofer & Pintrich, 1997; Muis, 2004; Sierpinska & Lerman, 1996). A person’s perception and understanding of what knowledge is, how individuals learn or acquire knowledge, how knowledge is transferred between entities, and how one can discern if others have learned inform the basis of that individual’s personal epistemological beliefs (Hofer, 2000; Muis, 2004; Sierpinska & Lerman, 1996). Research has shown that these beliefs influence a person’s intellectual performance (Kuh, Cheney, & Weinstock, 2000), learning, problem solving, mental effort, and academic success (Greene, Torney-Purta, & Azevedo, 2010; Hofer & Pintrich, 1997), as well as impact a teacher’s approach to teaching (Prawat, 1992; Speer, 2001; Thompson,
1984). These effects personal epistemological beliefs may have on members within the academic community led us to examine the development of personal epistemological beliefs.

Researchers found that several factors contribute to person’s epistemology development. These factors include age, education, life experiences, and teachers’ interaction with their students “Kuhn’s, Cheney’s, and Weinstock’s (2000); Speer (2001).” Among these factors, Kuhn’s, Cheney’s, and Weinstock’s (2000) argued that education appeared to be the most influential factor that shapes an individual’s epistemology.

Prawat (1992) suggested that instructors’ view of the transfer of knowledge informs their decision to teach their classes through transmission, where the act of teaching is viewed as reporting information to the students and students’ learning occurs through the absorption of this information (Kolitch & Dean, 1999), or student-centered instruction, where students are accountable for their own learning through classroom discussions and critical thinking activities that require students to construct their own understanding of concepts (Felder & Brent, 1996). This argument was echoed in Speer’s (2001) research on connection beliefs and teaching practices of teaching assistants in calculus courses. Speer found evidence that one of her participants expressed that his beliefs played an important role in his justifications for his teaching decisions and practices, specifically his constructivist belief that students needed to work independently and construct learning that they owned.

Thompson (1984) also examined the relationship between instructors’ beliefs and views on learning of mathematics and their teaching behavior. While Thompson discovered that these teachers’ beliefs and views significantly influenced their instructional practices, she also found other influencing factors as well. Thompson cited that teachers’ instructional practices were
informed by their knowledge of the mathematical curriculum and their knowledge of their students’ mathematical capabilities.

Learning Theories

The considerable supporting evidence for the influence of personal epistemologies on teachers’ instructional practices in the classroom causes us to examine the core learning theories found in mathematics education: (1) Behaviorism, (2) Cognitive Information Processing, (3) Situated Cognition, (4) Social Cognitive Theory, (5) Social Constructivism, and (6) Radical Constructivism. We will also consider the associated implications for mathematical instruction for each of these theories.

Behaviorism. Those following this understanding of knowledge and knowing typically explain learning in terms of environmental events that produce a positive or negative stimulus, and that these experiences form the basis for learning. It is also believed that this learning can only be assessed by one’s actions or behavior. This epistemology is seen in classroom management and repeated skills practice that allow for appropriate positive or negative reinforcement (Driscoll, 2005).

Cognitive Information Processing. This theory assumes that “input” information is taken from an individual’s environment, then processed and stored in one’s memory so that it may be retrieved as “output” to use learning new material or forming response. Teachers use this theory to inform their instructional practice in the organization of their instruction, creation of strategies that aid in students’ processing and storing of information, and ensuring their students properly process the information so that it may be available for future retrieval and use (Driscoll, 2005).

Situated Cognition. The belief that the main goal for learning is for individuals to produce meanings for their engagement in the social world through their interactions with
situations in a particular setting. Instructors who follow situated cognition view their students as apprentices; informing the instructors’ decision to focus on students’ development of skills for a particular situation so the students can apply these skills if encountered by a similar situation. Teachers will also stress students’ ability to communicate and collaborate with others so that they may contribute to the class’ learning (Driscoll, 2005).

*Social Cognitive Theory.* Followers of social cognitive theory believe an individual’s behavior is determined by a relationship between personal factors, behavior, and environment. They also see learning as something that occurs through doing a task or through observation of watching others do a task. These thoughts inform teachers believing in this learning theory to provide their students with models of appropriate methods or correctly worked example tasks for the students to model their own learning from. As the students model the teachers work, it is important to provide reinforcement to encourage desired behaviors (Schunk, 2004).

*Social Constructivism.* Advocates of social constructivism see knowledge as a product of social, cultural, and environment interactions, and not as something that occurs internally within an individual. Learning is believed to occur through collaborative efforts within a social setting, where individuals develop joint understanding of concepts and ideas (Ernest, 1996). Teachers using this belief typically see themselves as a supportive tool to assist their students’ construction of new knowledge by scaffolding students’ prior knowledge in collaborative and cooperative learning activities. The control of the dialogue in the classroom generally falls to the students, making them responsible for their own learning (Schunk, 2004).

*Radical Constructivism.* Similar to social constructivists, radical constructivists believe that an individual’s constructed mental world is the only reality, and this reality may only be true for that individual (Glasersfeld, 1995). This constructed mental world is developed as learning
occurs through disequilibrium and cognitive conflict (Tall & Vinner, 1981). Teachers create cognitive conflict for their students by posing questions slightly beyond the student’s current capabilities, scaffolding the student’s prior knowledge, and then use this conflict to make the student aware of the conflicts and misconceptions in his or her own understandings of a concept or idea (Schunk, 2004).

These core epistemologies typically serve as the basis for an individual’s understanding of knowledge and learning. However, believing in an aspect of one epistemology does prevent an individual from believing in an idea from another epistemology. Hofer (2000) argued that individuals may develop a personal epistemology that consists of beliefs from varying epistemologies. She also explained that individuals may alternate their epistemological belief depending on the situation that they find themselves.
Methodology

This methodology section outlines our research setting, population, sample, and theoretical framework. Subsections include discussions of the interview questions and role of the researchers. The Interview subsection provides an overview of the interview process adopted for this study, relating our research goals to the interview questions, and discusses the validity, reliability, and trustworthiness of the interview process. The Role of the Researchers subsection discloses any potential biases that may stem from our educational and instructional background, as well as our prior interactions with the participants of our study.

The site of this study was a public Doctoral Research university in the Rocky Mountain region of the United States. The Calculus I and II sequence at this university is divided by material covered. The first course focuses on differential calculus, implicit differentiation, and Newton’s method to approximate roots, while the second course involves mostly single variable integral calculus and convergence of series. At this institution the FTC is explored in Calculus II and serves as the connecting concept for both courses.

It is our radical constructivist philosophy that individuals construct their own subjective knowledge and interpretations of lived experiences (Creswell, 2007; Crotty, 1998; von Glasersfeld, 1995). Therefore, it is our perspective that even though our research population, university mathematics instructors that have experience teaching the FTC in undergraduate calculus courses, share similar educational and instructional experiences their individual perceptions and understanding of these experiences may vary. Since the mathematics instructors’ individual understanding of their epistemological beliefs and experiences teaching the FTC constitutes the focus of our study we adopted a phenomenological design for our research (Merriam, 2009).
Participants

In preparation for gathering participants for our research we first set out to discover which mathematics instructors at the university possessed experience teaching the FTC in an undergraduate Calculus course. Since the FTC is covered in the Calculus II course at the university of our study, we referenced the class listings for those instructors teaching Calculus II during the current spring semester, and we also questioned the mathematics faculty about other instructors that possessed experience teaching Calculus II. After we collected seven different instructors’ names, we solicited each instructor individually by email to participate in our study. Of these seven instructors that we contacted, four were willing to meet with individual members of our research team to be interviewed. These participants possess a varied educational and instructional background. Our four participants included:

Dr. Columbo: Dr. Columbo possessed a PhD in pure mathematics focusing on approximation theory and complex analysis. She has experience a wide range of undergraduate and graduate courses, including: different stages in the calculus course sequence, linear algebra, abstract algebra, basic analysis, real analysis, and complex analysis.

Hector: Hector possessed masters in mathematics with a focus on real analysis. He has been teaching courses in the Calculus sequence at the university of our research focus for just over 10 years.

Messi: Messi has almost four decades of teaching experience, primarily at the community college level. In particular, he has taught the calculus sequence numerous times. He has a master’s degree in applied mathematics and is in his second year as educational mathematics doctoral studies.
Patrick: Patrick was in his fourth and final year in an educational mathematics doctoral program. He possessed a strong mathematical background focused in combinatorics and modeling. Patrick also has diverse experience in teaching undergraduate introductory level mathematics courses, such as college algebra, applied calculus, calculus, and discrete mathematics.

Interview

This stratified purposeful sample for selecting our interview participants allowed for the comparison of characteristics and themes based upon the differences in the participants’ educational background and instructional experiences. Members of the research team met with the participants for one-on-one interviews. The interviews were conducted in a semi-structured format; audio recorded, and also transcribed (Creswell, 2007). This format for interviewing allowed the interviewer to direct the participants to discuss certain aspects of their educational experiences with mathematics so that we could later compare, contrast, and code the response data for reoccurring themes. This variation in interview participants strengthens the external validity of the research (Merriam, 2009).

The questions from each interview included some form of the sample questions listed in Appendix A, along with various follow-up questions. These questions were designed from our examination of the literature concerning the development of teachers’ PCK and epistemologies, and the impacts both have on instructional practices in the classroom. In addition to the literature, we also drew upon our own experiences and experiences of our peers regarding the teaching the FTC to refine our initial sample interview questions.
Role of Researcher

As researchers, we desire to be aware of and acknowledge any assumptions, bias, and theoretical or ontological perspectives that may influence our study (Merriam, 2009). Undue and unspoken bias could potentially undermine the objectivity and validity of the results of our study; thus, hampering the ability to generalize conclusions from my research (Crotty, 1998; Gall, Gall & Borg, 2003).

As researchers, we must also discuss our perspective of the importance of the FTC in undergraduate calculus courses. It is our belief as mathematics educators that the FTC is an important conceptual result in mathematics, and not simply a procedural tool, that should and could be understood by all calculus students, including its proof. By proof we mean a convincing argument for its truth, not necessarily the formality required by mathematicians and more advanced analysis courses since in this is the first exposure to the FTC the students are expected to experience.
Data Analysis

Trustworthiness and Dependability of Research

Throughout the duration of our study we continually sought consultation from both an expert, a mathematics educator with qualitative research experience, and peers, other members in the mathematics education graduate program, regarding the focus, purpose, structure, rigor, clarity, methods of analysis, coding, and presentation of our research. These discussion lead to the variation in our research participants, the structure of our interview process, the development of our audit trail of transcripts that allow for a peer to review our preliminary codes and themes from the interview data (Merriam, 2009).

Each member of the research team attempted to enter the interviews with our participants with a neutral, unprejudiced, open mind and remove any bias or influence our own understanding of (1) the FTC, (2) the nature of knowledge and its transfer between people, or (3) PCK could bring to the interviews or interpretation of the data. The foundation for our comprehension of these three aspects of our study resulted from our prior educational, instructional, and research experiences. While this background proved to be beneficial in the interview discussions related to mathematical concepts related to the FTC, students’ interactions with the FTC, and cognitive sciences in mathematics education, we remain aware of the potential of the unintended impacts on my research. Each interview transcript was peer reviewed by other members of the research team to ensure that an interviewer’s bias and experiences did not influence our overall findings.

To ensure we produced an authentic and accurate representation of our participants’ interview responses, we transcribed their interviews and e-mailed a copy of the transcript to each respective participant. This member checking (Merriam, 2009) allowed for our participants to
review their comments and make certain that the results from the interview reflected their true intended meanings.

Procedure for Analysis

While there exists extensive research concerning both personal epistemologies and PCK impacts on teachers’ instructional practices, there is a deficiency in the literature that specifically focuses on these impacts on the teaching of the FTC. Therefore, our study is an application of the existing research regarding personal epistemologies and PCK to explore the university instructor’s experience teaching the FTC in an undergraduate calculus course setting.

This exploratory quality led us to first individually read through our initial interview transcript (Hector) with an open mind, removing our assumptions and personal viewpoints. During this time, members of the research team would frequently stop reading to make notes in the text and the margins of the printed transcript, openly coding the interview participant’s responses for reoccurring themes and unexpected findings (Creswell, 2007). After this initial coding process, we began to compare our codes with the existing literature, and using the literature to inform the condensation and refinement of our initial codes. Once we established our revised axial codes, we created a coding glossary (see Appendix B) that provided a description of each of our codes and an example of the code found in the data from Hector’s interview responses. These marked up copies of the interview transcript and coding glossary serve as an audit trail that can be referred to understand how we arrived at my results or findings of the data (Merriam, 2009).

During the development of our coding glossary, our research team would meet to discuss and argue each of our individual perceptions on the coding scheme resulting from Hector’s interview transcript. Through these discussions and arguments, our group came to consensus on
the coding scheme for Hector’s interview and the definitions for each of the codes used in the scheme. Then we took the result of our comparative coding process and created a final coding scheme using NVivo, a qualitative coding software package.

The codes and coding scheme from Hector’s interview served to inform our initial coding of the other participant interviews (Dr. Columbo, Patrick, and Messi). We again started my coding process by making notes in the text and margins of a printed copy of the participants’ interview transcripts. While we applied codes that were developed from Hector’s interview in this coding process, if there was a theme in a participant’s interview that did not fit under these codes we created a new code for the theme. After reviewing our audit trails for our other three participants’ interviews to check our understanding of the codes, we reread the transcripts to establish a more refined and condensed set of axial codes from our initial coding and then recoded the interview in NVivo. Following this, we again reviewed our axial codes in attempt to refine and condense them into major themes of the interview data.

Once we completed polishing our codes and themes, we then developed significant statements from these themes and attached the participants’ responses that we identified and interpreted to reflect these statements in the clearest and most efficient matter. Finally, we compared our resulting statements and themes with the literature to determine if our findings aligned, or if there is an unaccounted case that may be an area of interest for future research.
Findings and Discussion

*Epistemological stance*

In general, it was not possible to classify participants unambiguously with respect to epistemological stance. All participants used elements of different viewpoints in their teaching. Messi, for example, cannot be easily labeled or categorized as adhering to any of the main viewpoints. He thinks that “the students are different” and “every student learns their own way.” He resists delineating or categorizing his stance in the abstract: “It’s hard to tell you some generalization of this.”

In descriptions of his own teaching practice, Messi exhibits tendencies compatible with several epistemological viewpoints. Much of the time he appears to be adhering to a behaviorist-style, transmission-of-knowledge mode of teaching. This includes “show[ing] a lot of examples,” for instance. His way of dealing with student mistakes on exams is another typical example of a behaviorist approach to teaching:

> Every exam I analyze it to look how they responded, to look at the questions that had good responses and the questions that had bad responses in general, so I repeat those questions and I emphasize on the exam at the end, giving them solutions and discussing the exam completely.

He even says, “I like to lecture of course, that’s my favorite thing.” But then he quickly qualifies this statement with a decidedly more constructivist complement:

> However, it’s not the only thing that I do. I always like to expose a problem at the beginning of class and tell them to talk to each other, give them a couple of minutes to discuss this thing and ask them: please do something, write something
on your paper, what would you do? Or, How do you feel now? Whatever it is write it down since this is not like an exam or anything.

This approach also agrees well with Messi’s emphasis on differences in learning styles. Another such situation occurred when Messi was asked about a hypothetical scenario in which a student was trying to integrate a function with no elementary antiderivative. Here Messi says he would try to “stimulate curiosity” by telling the student about how “eventually we will discover power series,” “and then you will be able to integrate term by term, and then you can come up with something close.” Messi also deals with errors in reasoning in a similar spirit:

I always like to approach the students as that, not that they’re wrong. Let’s think about it again. Why do you think it’s that way? Instead of saying that’s wrong or there’s a mistake in there.

Misconceptions are “something that they have to overcome themselves” and “I’m there to help,” is Messi’s attitude. It is important to keep in mind, however, that Messi’s approach to teaching is far from exclusively guided by epistemological considerations. For example, he says:

What I do all the time also is to review. Reviewing for me is always very important.

But the reasons for this are perhaps not primarily epistemological:

I know that some students are not as well prepared as others. But that will take of itself later so that eventually if they cannot handle it they will drop the course by themselves but I don’t want to scare them the first day. Maybe they will realize that they are just not well prepared and then they will take a decision to drop the class but I don’t want to … [be] the bad guy who … you know, scare them off.
One may speculate that perhaps Messi’s background at community colleges, with their often pragmatic view of education, has led him to place emphasis on pragmatic rather than epistemological aspects of teaching.

It is interesting to compare Messi’s case with that of Patrick, who had a very different background. As an advanced doctoral student in mathematics education, Patrick was well versed in learning theories and their potential applications in the classroom. Interestingly, however, he too emphasized that he did not believe that there was one correct learning theory; rather, he drew upon several different traditions in his teaching:

I think there are many ways to learn in mathematics, I think it’s a combination of social interaction and sort of individual thinking about topics that happens both while you are doing mathematics …. [But] I should say that mostly they learn mathematics by interacting with it, so it is very hard to learn mathematics by reading it in the book for example.

The behaviorist side of Patrick appeared when he was talking about his students and the things that he is more concerned about while they learn mathematics:

I mean ultimately I’m more concerned with them working on mathematics than understanding mathematics. So, I know it sounds some kind of weird. I’d rather that they are spending a lot of time trying to understand mathematics, and I am sure that they will understand some of it on their way but it is not entirely important to me that they will become masters at each of the things that we do.

Like Messi and Patrick, Dr. Columbo also emphasized that she believed that “…our brains are very different, and that there are different paths to learning.” She based this belief on her experience interacting with others’ learning, “I’ve met people who honestly seem to bypass
having to construct as much [knowledge] and just have flashes...” and her own learning experiences, which she described as a that of a constructivist.

Dr. Columbo asserted that her understanding of knowledge and learning was framed as that of a constructivist. There is evidence of the influence of her self-professed constructivist epistemology in her teaching of the FTC, such as in her scaffolding of a student’s prior knowledge of area to help the student construct new knowledge concerning the integral of and in her use of group learning. However, it appeared that Dr. Columbo utilized or followed epistemological ideals in her instructional practices other than constructivism as well; aligning with Hofer’s (2000) argument that individuals may develop a personal epistemology that is a blend of beliefs from the core epistemologies. Dr. Columbo appeared to follow ideas from behaviorism in her perception that it is impossible to “know what’s going on in their [students] heads...” and that she seemed to assess their knowledge by observing the students behavior and work. Dr. Columbo’s belief that students learn through “lots of examples” is evidence of the use of situated cognition, as the examples serve as a model of skills Dr. Columbo desires her students to master.

In contrast to the other participants, Hector had a quite distinct epistemological stance, which may be characterized as behaviorist. This is seen most clearly in how he makes the students learn the statement of the FTC:

Every day the class as a whole recites the damn [FTC] and they get pretty good after a while. I can just stand there and look at them and say “ok, begin” and they all just start saying it. It is strictly rote.

Hector’s way of dealing with common errors is also behaviorist in flavor:
And in lectures I will demonstrate … I’ll show them common errors and then we will go back and check it: here’s where things went bad, this is a common thought, don’t do it, here’s why.

To this, however he adds an interesting qualifier which raises doubts as to whether his behaviorist leanings are motivated by convictions regarding student learning:

As whether or not it helps, I don’t know. It makes me feel good.

But this should perhaps be taken more as an off-the-cuff joke, for when asked “How do you believe students acquire knowledge in mathematics?” Hector answered without hesitation:

Examples. That’s it. Examples. I hand them definitions, I hand them theorems, and then I give them a whole mess of examples.

We should note, however, that when Hector is describing his epistemological stance in abstract terms he does not sound as behaviorist as in his accounts of his teaching. For example, he says:

My teaching style is not constructivist, but I will use constructivist techniques […]

I don’t believe I can teach anybody anything—I don’t think anybody can ever teach with that idea of: open head, pour in facts, shake well. I don’t think you can do that.

Overall, our findings are in agreement with results in the literature to the effect that age, experience, and education are factors in an individual’s epistemological development (Kuhn, Cheney, & Weinstock, 2000), and that epistemological considerations influence instructional practices (Muis, 2004; Speer, 2000; Thompson, 1984). However, we also found that epistemological considerations are rarely clear-cut and that they are highly context-dependent. Thus our findings suggest that it would be misleading to think of a teacher’s epistemological
stance as determinate factor shaping his or her teaching; rather epistemological considerations evolve in tandem with pedagogical and content-specific considerations.

**Pedagogical content knowledge**

An important aspect of PCK is knowledge about common student mistakes and misconceptions. Our participants’ knowledge in this area was high and uniform; their replies were knowledgeable and very similar when presented with hypothetical examples of student reasoning. This is perhaps not surprising seeing as they all had significant teaching experience in the area.

Despite this overall uniformity with respect to PCK, we found some evidence that the participants’ epistemological stance had a noticeable impact on their thinking in this regard. For example, Hector’s overall behaviorist stance seemed to shape how he thought about student errors in that he thought of students as failed behaviorist subjects following the wrong rules. When asked about the most common error he immediately replied:

Algebra skills. Oh geez. They just love canceling. And they make up new rules. ...

They love to make up new rules where they are just manipulating symbols and not concepts.

By contrast, Patrick, who had a more constructivist view, emphasized more conceptual issues:

The prerequisite for the student to be able to appreciate this … They have to have a very strong understanding of functions to really get the theorem. […] The idea that you could define a function by putting a variable on the top of the integration symbol is very unusual; this is very strange for them. So, I am anticipating that is being weird.
This discrepancy of conceptions of student mistakes vary among teachers is reflected in the literature. For example, Thompson (1994) maintains that “students’ difficulties with the [FTC] stem from impoverished concepts of rate of change” (p. 229), whereas Foley (1992) and Mahir (2009) attribute it to lacking conceptual understanding of integration. Thomas (1996) in turn maintains that the crucial factor is to understand functions (including functions defined as integrals) at the object level of the APOS hierarchy (esp. p. 89).

We believe that our analysis, which places these discrepancies in the context of epistemological theories, can be a useful synthesizing perspective for studying these issues. However, since this was an emergent finding in our research, further research is needed to investigate this possibility in more depth.

Furthermore, not only knowledge of student mistakes is an important part of PCK; equally important are positive aspects, i.e., what students find convincing and interesting and stimulating. No research has been done on this aspect of PCK in the case of the FTC. Our study, as well as that of Klisinska (2009), found little agreement among calculus teachers regarding how they introduce and motivate new material to the students. This is also a potential topic for future research.

Curricular content knowledge

Issues of curricular organization occur naturally when discussing the FTC, since it is typically discussed in two courses: first from a computational point of view in calculus and then later from a more theoretical point of view in real analysis. All participants took note of this fact, often noting that a formal proof of the FTC did not belong in a calculus course. However, different participants had different ways of dealing with this fact.
Hector and Messi, the two participants with the most extensive teaching experience at the introductory undergraduate level, both chose not to prove the FTC, referring to later courses in analysis. Hector said for example:

I never prove the evaluation theorem. … The mathematician in me says I better back off for a proof. Although I do get all the pieces in, where we talk about the necessity for continuity... I don’t know that you can prove that without epsilon and deltas. … I think actually proving that is really better suited for a 500-level analysis course.

Messi’s approach was similar to that of Hector. We noted above that Messi’s teaching seemed to be influenced more by pragmatic factors than epistemological ones. Another aspect of this pragmatic approach to teaching is a conception of the material taught that is very closely tied to its curricular organization. An example of this is Messi’s attitude towards proving the FTC. The traditional curricular division of labor between calculus and analysis is of course to teach computation in calculus and theory in analysis. Messi seems to be happy with this division and says that he “never proved [the FTC] in class,” referring instead to later courses in real analysis. The downside of this division of labor between calculus and analysis might be that perhaps an intuitive approach to understanding why the FTC is true is precluded by the presence of “the” proof in later courses in analysis. Something like this was suggested by the interviewer, who agreed that “it is not necessary to give a very rigorous mathematical proof” but suggested that instead “some intuition thing … to convince the students” might still be useful. But Messi did not appear receptive to such an approach. This could be seen as a form of deficient PCK in that the conceptions of proof are constrained by the curricular organization of the material and one may hypothesize that a non-research mathematician would have his thinking more closely tied to
curricula, which would agree with the present cases. Hector, similarly, appeared to be belittling the importance of an intuitive, conceptual grasp of why the FTC is true when he characterized the theorem in these terms:

In some sense, slope and area are the same thing, in some weird, ludicrous, only-a-mathematician-can-appreciate-it kind of way.

The attitudes of Messi and Hector agree with the findings of Klisinska (2009), who reported agreement among her participants that students are “interested mainly in the computational aspects of the FTC, as well as generally in their mathematics studies showing a procedural approach rather than a conceptual” (p. 119).

This shunning of the theoretical and conceptual aspect of the FTC was avoided by Dr. Columbo, who took a somewhat different approach to dealing with the divide between the calculus and analysis courses. Her CCK appeared to be directly informed by her past teaching experiences as an instructor for Calculus I, Calculus II, undergraduate and graduate Basic Analysis, and graduate Real Analysis, as she refers to the application of the FTC to discontinuous functions as an “analysis issue”. Columbo goes on to explain that in her undergraduate calculus courses she focuses on “continuous functions, that’s it. I’m not trying to do anything else”, providing evidence that her CCK influences her decision to teach the FTC primarily with continuous functions in her undergraduate calculus courses.

Dr. Columbo also demonstrated her CCK in her knowledge of a Calculus curriculum from varying textbooks from her past experiences teaching the FTC; in particular the “Thomas and Finney, the standard Calculus books”, that involves the partitioning of the area under a function curve using the Riemann sums, and from “Harper Calculus”, that involves the considering the derivative of the area function of a rectangle under a curve. She uses this knowledge to inform
her decision to use the intuitive geometric approach for teaching the proof of the FTC, as discussed in the Harper Calculus book, citing that she “found it [Riemann sum proof] particularly unsatisfying, it [Riemann sum proof] sort of minimizes the importance of it [FTC].” Thus, providing further support that CCK plays a role in Dr. Columbo’s teaching approach to the FTC. This decision to present the proof of the FTC using the intuitive geometric approach in her teaching is also informed by her past experience teaching interactions with students. She describes that only “rare students” could understand the formal teaching of the proof, and this reasoning leads her to present the intuitive geometric proof.

However, Dr. Columbo also noted that students lacked motivation to conceptually understand the FTC. She described “students that already know how to find how to evaluate integrals and they’re thinking why is she up there talking about why this works, I could care less.” She then pointed out the lack of assessment tasks involving the importance of understanding the FTC. Dr. Columbo described problems typically associated with the FTC section as “silly because nobody [students] seems to know what they do” and that these problems are “actually not in any way assessing their understanding of the [FTC]”. She then acknowledged that “I don’t feel like there is any good assessment” that would inform her knowledge of students’ mathematical understanding of the FTC and stress the importance of the FTC to the students.

In conclusion, we found that the traditional curricular division of topics between calculus and analysis courses was an important factor influencing teaching styles, and that the participants considered this division to be appropriate on the whole. This is in agreement with the results reported by Klisinska (2009). However, one of our participants expressed concern that the traditional curricular division of topics hinders the implementation of a more conceptual and
exploratory approach when the FTC is first introduced. We found some evidence that these considerations mirrored epistemological ones, with more behaviorist-oriented teachers being more content with the traditional emphasis on computation in calculus courses.

*Specialized content knowledge*

All participants had solid specialized content knowledge, which informed their teaching in a number of ways.

One use of SCK was to employ it for pedagogical purposes. Patrick, for example, when explaining the different aspects of the FTC and its applications and relations to other aspects of mathematics, provided several examples from different analytical points of view. Also, he explained in detail how to relate integrability to continuity in the FTC and overcome some students’ misunderstandings. Thus his strong specialized content knowledge allows him to better provide his students with counterexamples and different ways to think about the same problem in order to help them appreciate the theorem.

Well I think both of them could learn from looking at a picture and thinking about what it means to integrate something that … is finding the area of it or the accumulation of it. I mean I think it depends on their major whether they should think of it as a geometry problem or as accumulation problem. If they are science major it should be accumulation, and if they are math major then geometry is natural. So, anyway I think both of them could clear up their two questions by looking at a picture of it, what can be done here.

Another use of SCK was to convey enthusiasm and stress the importance of main ideas. For example, Dr. Columbo’s SCK impact on her enthusiasm for the FTC is evident in her description of the importance of teaching the FTC, “it connects the two huge ideas of calculus. It’s an
absolutely amazing result. I think it’s still amazing to this day.” Her SCK also informs her of the importance of the FTC and its applications in Calculus, which then influences her teaching of the FTC. Dr. Columbo stresses to her students her perceived importance of the FTC by stressing that “any theorem that says fundamental you gotta believe it’s big” and explaining that “it’s the theorem that connects the two parts of the course”. However, she acknowledges that her students often do not value the FTC as highly as she does, and primarily focus on the resulting applications of the FTC.

Similarly, Hector also used his SCK primarily when trying to motivate good students. When posed with a hypothetical scenario of a student trying to integrate \( \sin(x^2) \), Hector said he would tell an average student that “this is a wicked hard problem” and not to worry too much about it. But “if it were a really good student” he would utilize his SCK to stimulate interest:

I’d turn to Mathematica and say: let’s have Mathematics chew on it. And then it would turn around and give me something relating... error function \(-e^x\)? And he’d say: error function? And I’d say: well, you know, nobody knows how to anti-differentiate this but it turned out that it was important so we made something up.

In conclusion, we found that our participants demonstrated informing connections between pedagogical knowledge and subject matter knowledge. The interconnections that we found are schematized in Figure 1. Thus our findings call into question Hill’s, Ball’s, and Schilling’s (2008) concept of PCK and subject matter knowledge as consisting of disjoint, mutually exclusive subsets of teacher knowledge. However, as it was not the primary purpose of this study to revise these theories, further study of the nature of the interconnections suggested by our results is left for future research.
Limitations of the Study

The sample of the study was chosen from one institution (convenient sampling). However, the research questions in this study could be investigated using a larger sample by including more institutions. PCK is still a young field. Therefore, this study could be extended to study more topics in calculus in particular and analysis in general such as differentiation, integration, sequences and series.

A limitation of this study is the sampling method. The small size of the population did not allow using randomization. Also, we contacted seven instructors and four of them agreed to participate only one of them was a female. This resulted in a gender bias in the sample of the study. By the time we started collecting our data, the instructors were already finished teaching the FTC which prevented us from doing some field observations to triangulate our data. Also one of the limitations of the study is that we conducted the study in only one semester which is a relatively short time to come with generalizable results.
References


Appendix A

Sample Interview Questions

Participant Profile Information:

1. What is your instructional background?
2. What is your educational background?
3. What is your perception of what knowledge is?
4. How do you believe students acquire knowledge in mathematics?
5. How does your own educational background inform your approach to the Fundamental Theorem of Calculus?

Mathematical Knowledge for Teaching:

1. What is the importance of teaching the FTC?
2. How do you introduce the FTC for the students?
3. What knowledge or skills do you draw upon to understand and apply the Fundamental Theorem of Calculus?
4. Explain the types of prior knowledge and skills that you believe are essential to students’ understanding and application of the Fundamental Theorem of Calculus.
5. How do these tools aid in students’ comprehension of the Fundamental Theorem of Calculus?
6. Can you explain the obstacles students encounter learning the FTC? If there are none, can you explain why you believe this?
7. What major types of mistakes (other than computational errors) or difficulties have you observed from your students when they apply the FTC?
8. In your teaching of the FTC, what importance do you place on the continuity of the integrand?
9. What importance do you place on the proof of the Fundamental Theorem of Calculus?
10. In his homework, John tried to find an elementary anti-derivative for the function \( \sin x^2 \), John started from the fact that this function is continuous, what comments would you give to him to help him understand the existence of the anti-derivative and computing it?

11. Below is an example of how a student calculated the integral \( \int_1^e \ln x \, dx \)

\[
\int_1^e \ln x \, dx = \left. \frac{1}{x} \right|_1^e = \frac{1}{e} - 1
\]

a. What might he be thinking when he decided to calculate the integral in that way?
b. What comments would you give to him to correct his mistake?

12. The following conversation took place between two students:

Lindsey: Can I apply the FTC for \( f(x) = \begin{cases} x, & -1 \leq x < 2 \\ x^2, & 2 \leq x < 7 \end{cases} \) on the interval \([-1, 7]\)?

Frank: No you can’t.

Lindsey: Why?

Frank: In order to apply the FTC, the function \( f(x) \) must be continuous on the interval.

Lindsey: The FTC says: if \( f(x) \) is continuous, then you can find the anti-derivative but what if \( f(x) \) is not continuous?

a. What might each of the student’s be thinking?
b. Is Lindsey’s answer wrong? Why? Or why not?
c. Why, do you think, Lindsey has this kind of question?
d. What can be done to overcome the students’ misconceptions about the FTC?
e. What kind of approach can be used to correct his understanding?
f. What kind of application(s) can be provided to help them?
Appendix B

Coding Glossary

Transmission - (Kolitch & Dean, 1999) – In the transmission model, the act of teaching is seen as imparting information and learning is taking in or absorbing information. Direct transfer of knowledge from instructor to student.

Example: “I hand them definitions, I hand them theorems, and then I give them a whole mess of examples.” (Hector)

Radical Constructivism - An individual’s constructed mental world is the only reality, and this reality may only be true for that individual (Glasersfeld, 1995). This constructed mental world is developed as learning occurs through disequilibrium and cognitive conflict (Tall & Vinner, 1981). Teachers create cognitive conflict for their students by posing questions slightly beyond the student’s current capabilities, scaffolding the student’s prior knowledge, and then use this conflict to make the student aware of the conflicts and misconceptions in his or her own understandings of a concept or idea (Schunk, 2004).

Example: I know that some people believe that, that knowledge that you construct you feel better... you have better ownership. (Hector)

Behaviorism - Learning is explained in terms of environmental events that produce a positive or negative stimulus, and that these experiences form the basis for learning. It is also believed that this learning can only be assessed by one’s actions or behavior. This epistemology is seen in classroom management and repeated skills practice that allow for appropriate positive or negative reinforcement (Driscoll, 2005).

Example: “My desire is that, and what I try to get the students to do, is for example, I’ll show them how to make chicken soup and then I’ll show them how to make hot-buttered noodles, and then I stand back and say: “Ok, make chicken noodle soup!”” (Hector)

Rote - Learning technique which avoids understanding of a subject and instead focuses on memorization. The major practice involved in rote learning is learning by repetition. The idea is that one will be able to quickly recall the meaning of the material the more one repeats it.

Example: “And we do this every day, and every day the class as a whole recited the damn fund... and they get pretty good after a while.” (Hector)
**Knowledge of Content and Students (KCS)** - Knowledge of students’ mathematical thinking and learning (Hill, Ball, & Schilling, 2008, p. 373). Content knowledge intertwined with knowledge of how students think about, know, or learn this particular content (p. 375).

*Example:* “The new rule that the students love to make up is the one that says: oh hey, one over stuff, anti-derivative natural log stuff. No. They love to make up new rules where they are just manipulating symbols and not concepts.” (Hector)

**Past Experience** - Discussion of instructor’s previous teaching experiences. A subcategory for KCS.

*Example:* “It is strictly rote. It’s how I approach it. But I’ve tried in the past where I don’t lose my freaking mind over it and they just... they don’t care.” (Hector)

**Curricular Content Knowledge (CCK)** - Substantive subject matter knowledge about topics, procedures, and concepts along with a comprehension of the relationships among them as they are offered in school curricula (Hauk, Kreps, Judd, Deon & Novak, 2010).

*Example:* “But... hopefully John can do the chain rule. If not I’d probably begin weeping. How did he get into Calc II?” (Hector)

**Specialized Content Knowledge (SCK)** - Mathematical knowledge beyond that of a well-educated adult, but does not require knowledge of students or teaching (Hill, Ball, & Schilling, 2008).

**Conceptual Connection** - The instructor makes an explicit connection between derivatives and anti-derivatives using the FTC.

*Example:* “The FTC does two things. It is (1) a recipe for how to evaluate a definite integral, and (2) it shows the relationship between derivatives and what we call anti-derivatives.” (Hector)

**Enthusiasm** - Instructor conveys an increased level of interest or awe concerning a particular matter.

*Example:* “As I’m talking about this [FTC] one of the things that is important is that I try to convey my own enthusiasm. That this concept is not important but that at least one person on this planet thinks it’s interesting.” (Hector)