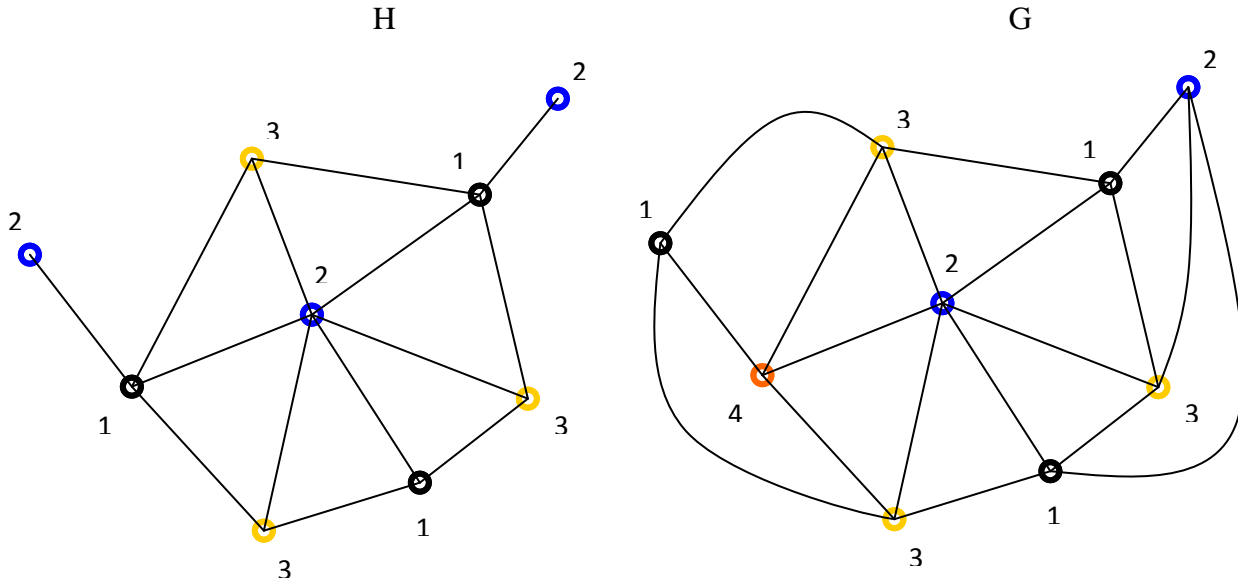


**Theorem:** All planar graphs  $G$  can be 5-list colored:  $\chi_l(G) \leq 5$ .

Before proving the theorem, there are a few things worth noting. Observe that if you add edges to a graph it can only increase the chromatic number. Consider the example of the subgraph  $H$  of our graph  $G$ .



Clearly, adding the five edges to the subgraph  $H$  could possibly force us to reconsider our coloring and require more colors to color our graph  $G$ . Therefore, we see that  $\chi_l(G) \geq \chi_l(H)$ . Thus we may assume that  $G$  is connected and that all the bounded faces of an embedding have triangles as boundaries. The graphs of this form are known as near-triangulated graphs.

Now for the proof of the *5 Color Theorem*.

**Proof:**

The key to this proof lies within the following assumptions for  $G = (V, E)$ , where  $G$  is a near-triangulated graph, and  $B$  is the cycle bounding the outer region of graph  $G$ . We make the assumptions on the color sets  $C(v)$ ,  $v \in V$ :

- 1) Two adjacent vertices  $x, y$  of the boundary  $B$  are already colored with different colors  $\alpha$  and  $\beta$ .
- 2)  $|C(v)| \geq 3$  for all other vertices  $v$  of  $B$ .
- 3)  $|C(v)| \geq 5$  for all vertices  $v$  in the interior.

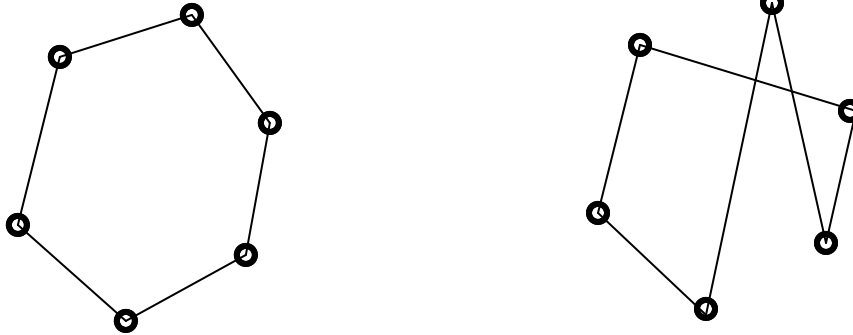
Then the coloring of  $x, y$  can be extended to a proper coloring of the graph  $G$  by choosing colors from the lists. In particular  $\chi_l(G) \leq 5$

These assumptions will allow us to determine our coloring inductively.

Recall the following definition:

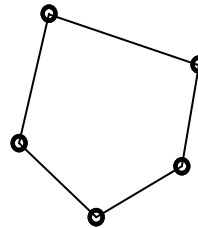
**Definition:** A *cycle* is a closed path with no repeated vertices or edges other than the starting and ending point of the path.

Examples:



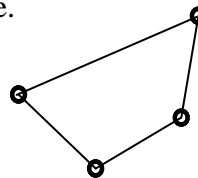
Consider our graph  $G$  that consists of only three vertices, say  $x$ ,  $y$ , and  $v$ . Notice that the only uncolored vertex  $v$  we have  $|C(v)| \geq 3$ , so there is a color available for the vertex  $v$ .

Now construct your own cycle, make it as complicated or as simple as you wish. Then color your cycle using the least number of colors possible.



The number of vertices used to create your cycle was 5.  
 It would take 3 colors to color the cycle.

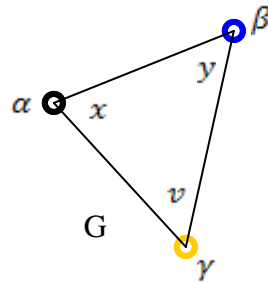
Now take away one of the vertices in your cycle. Create a new cycle with these vertices. Then color this cycle with the least number of colors possible.



The number of vertices used to create this cycle was 4.  
 It would take 2 colors to color the cycle.

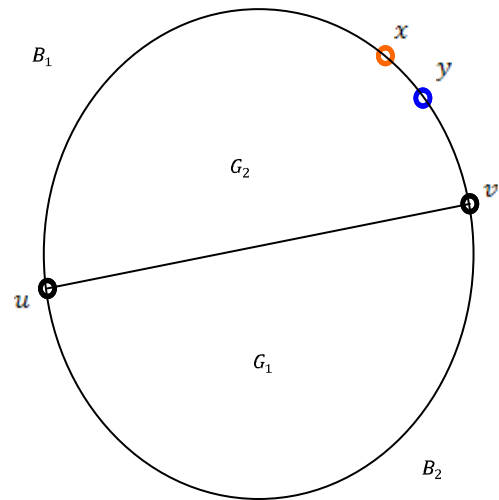
Therefore, we see that if we have an odd number of vertices it would take 3 colors to color our cycle and if we have an even number of vertices it would take only 2 colors to color our cycle.

Now for the first inductive step in our proof.

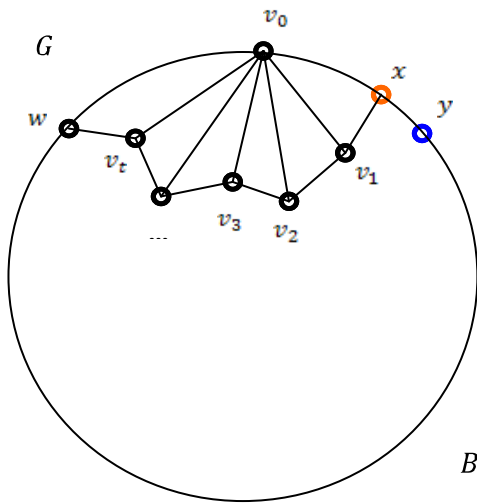


Using our assumptions we will continue by induction.

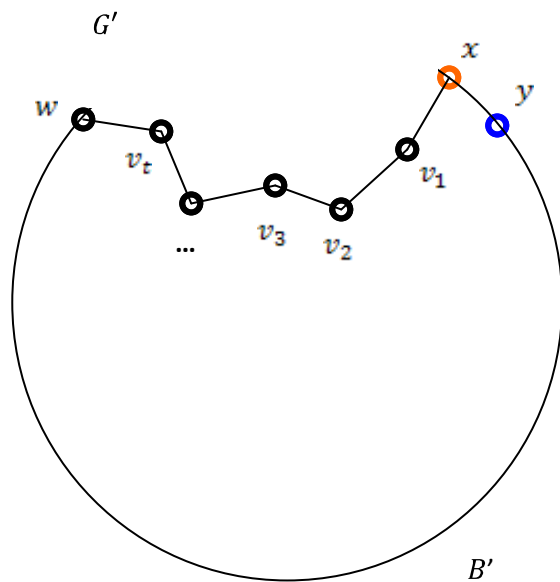
**Case 1:** Suppose  $B$  has a chord, that is, an edge not in  $B$  that joins two vertices  $u, v \in B$ . The subgraph  $G_1$  which is bounded by  $B_1 \cup \{uv\}$  and contains  $x, y, u,$  and  $v$  is near-triangulated and therefore has a 5-list coloring by induction. Suppose in this coloring the vertices  $u$  and  $v$  receive the colors  $\gamma$  and  $\delta$ . Now we look at the bottom part  $G_2$  bounded by  $B_2$  and  $uv$ . Since  $u$  and  $v$  are pre-colored  $\gamma$  and  $\delta$  by our work with  $G_1$ , we can use the induction hypothesis for  $G_2$ . Therefore,  $G_2$  can be 5-list colored with the available colors. Hence the same is true for our entire graph  $G$ .



**Case 2:** Suppose  $B$  has no chord. Let  $v_0$  be the vertex on the adjacent to the  $\alpha$ -colored vertex  $x$  on  $B$ , and let  $x, v_1, v_2, \dots, v_t, w$  be the neighboring vertices of  $v_0$ . Observe the following graph  $G$ .



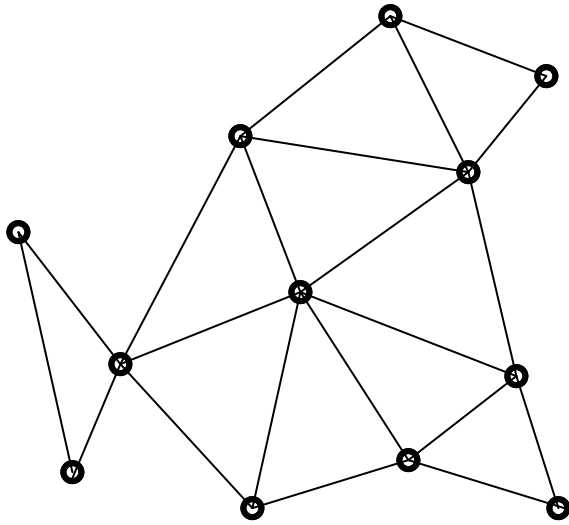
Notice that  $G$  is near-triangulated, Construct the near-triangulated graph  $G' = G \setminus \{v_0\}$  by deleting the vertex  $v_0$  from the graph  $G$  and all the edges connected to  $v_0$ .



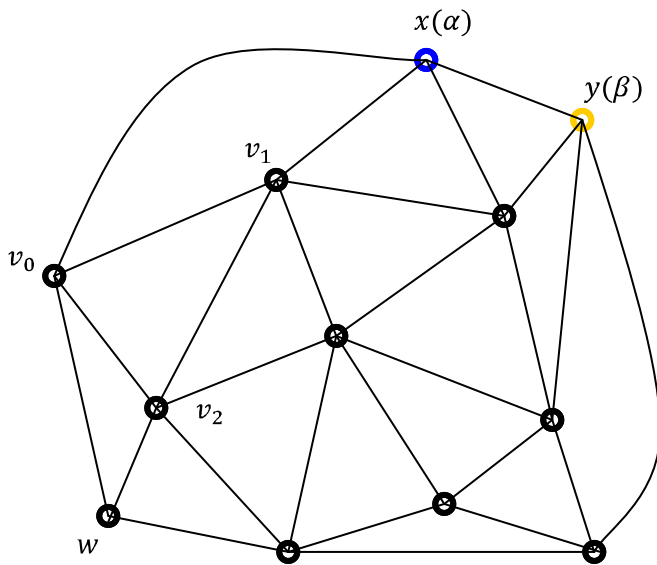
This  $G'$  has an outer boundary defined by the following  $B' = (B \setminus \{v_0\}) \cup \{v_1, v_2, \dots, v_t\}$ . We know by our second assumption that  $|C(v_0)| \geq 3$ , and that there must exist two colors  $\gamma$  and  $\delta$  in  $C(v_0)$  that are different from  $\alpha$ . Now we want to remove the colors  $\gamma$  and  $\delta$  from the color sets for each of the vertices  $v_1, \dots, v_t$  since we know that  $v_0$  will have one of those two colors. Thus, we can think that if  $C(v_i)$  was the original color sets for the vertices  $v_1, \dots, v_t$ , their new color sets will be of the form  $C(v_i) \setminus \{\gamma, \delta\}$ . The remaining vertices in  $G'$  that did not share a common edge with  $v_0$  may maintain their original color sets in  $G'$ . Now  $G'$  satisfies all three of our assumptions, and is

therefore 5-list colorable by induction. Then we can return to our graph  $G$  and choose  $v_0$  to be color  $\gamma$  or  $\delta$ , a color different from  $w$ , and show that  $G$  must also be 5-list colorable. ■

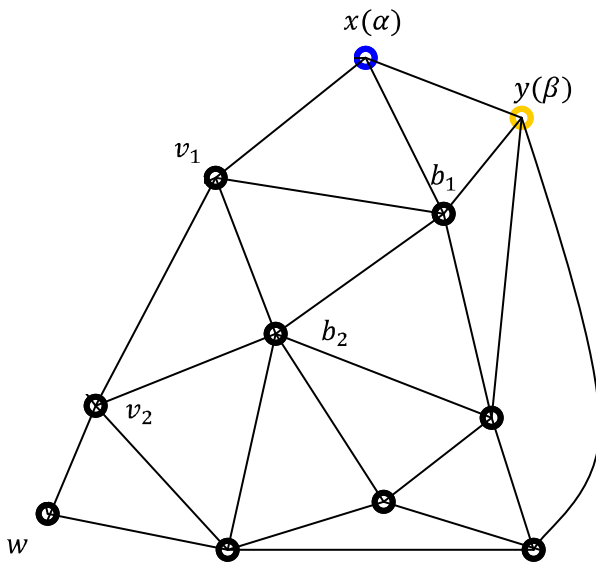
We will attempt to use this *5 Color Theorem* to help us create the coloring for some graph. Suppose that we have the color set  $\{\alpha, \beta, \gamma, \delta, \varepsilon\}$  to color our graph. Consider the following graph shown below:



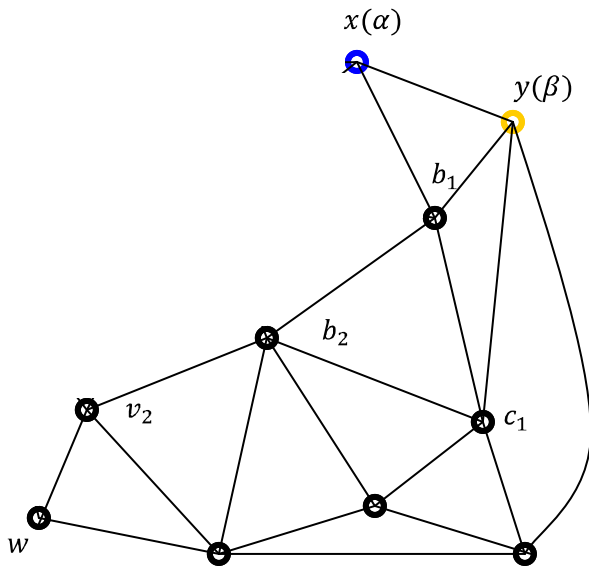
We will attempt to use the same ideas from the proof to create our coloring. Notice that all the vertices in this graph are not connected. Therefore, our first step is to connect all the point in our graph to make our graph near-triangulated.



Notice that in our near-triangulated graph we have no chords connecting any two vertices on the boundary. Therefore, we will be following **Case 2** of our proof. Denote two adjacent vertices on the boundary as  $x$  and  $y$ . Then color  $x$  with  $\alpha$  and  $y$  with  $\beta$ . Then name the other boundary vertex adjacent to  $x$  as  $v_0$ . Then we can label the interior vertices connected to  $v_0$  as  $v_1, v_2$  and the adjacent boundary vertex as  $w$ .



Now we remove the vertex  $v_0$  from our graph and all the edges connected to it. We then allow  $v_0$  to have the possibility of being either color  $\gamma$  or  $\delta$ . Then we know  $v_1$  cannot have color  $\alpha$ ,  $\gamma$ , or  $\delta$ , which leaves  $v_1$  can have color  $\beta$  or  $\epsilon$ . Similarly,  $v_2$  can only the choice of colors  $\{\alpha, \beta, \epsilon\}$ . Now we will repeat the process for the vertex on the boundary adjacent to  $x$ , which is not  $y$ . So now we will remove  $v_1$  and all the edges connected to it.



We can denote the interior vertices connected to  $v_1$  as  $b_1$  and  $b_2$ .

Notice that now our choices for colors for the vertex  $b_1$  come from the set  $\{\gamma, \delta\}$ . Also, our choice of colors for  $b_2$  are now  $\{\alpha, \beta, \epsilon\}$ .

Observe that we now have entered into *Case I* of the *5 Color Theorem* since we have the chord  $b_1y$  on the boundary of our subgraph. This ensures that we continue to satisfy the assumptions necessary to use the inductive step from the proof above. Therefore, we know that our new subgraph can be colored using at most 5 colors.

Since all we want to use the idea from the proof of the theorem to help us with the coloring, we will continue to deconstruct the graph to eliminate color combinations for vertices. So we once again repeat our process for the vertex on the boundary adjacent to  $x$ , which is not  $y$ . So now we will remove  $b_1$  and all the edges connected to it.



Thus we see this graph can be colored with our five colors  $\{\alpha, \beta, \gamma, \delta, \varepsilon\}$ . Therefore, we know our original graph can also be colored with only 5 colors.

