SUMMATION NOTATION AND PATTERNS

There are many situations where we may need to sum or add a large number of numbers. Since this happens frequently, mathematicians have adopted notation, called *summation notation* or *sigma notation*, to use in these situations.

An **index variable** is used to number or label each term in the summation.

The **start** and **end** numbers indicate the first and last terms of the summation.

The **formula** describes the form of each term in the summation.

To expand or rewrite sigma notation as a sum, we substitute each value of the index variable into the formula separately (from the start value to the end value) and then sum all of these terms.

To help understand sigma notation we will go through a couple examples.

**Example 1**

\[
\sum_{j=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55
\]

**Example 2**

Given the formula \(a_j = (j - 1)^2\) we see that

\[
\sum_{j=0}^{7} a_j = \sum_{j=0}^{7} (j - 1)^2 \\
= (0 - 1)^2 + (1 - 1)^2 + (2 - 1)^2 + (3 - 1)^2 + (4 - 1)^2 \\
+ (5 - 1)^2 + (6 - 1)^2 + (7 - 1)^2 \\
= 1 + 0 + 1 + 4 + 9 + 16 + 25 + 36 = 92
\]
Now consider the following summation:
\[ \sum_{k=1}^{9} a_k \]

How many terms are there in this summation?

**WE WILL SUBSTITUTE 1 THROUGH 9 INTO THE FORMULA \( a_k \), SO THERE WILL BE 9 TERMS.**

Given the formula \( a_k = 2k - 1 \) expand and simplify the summation

\[
\sum_{k=1}^{9} a_k = (2(1) - 1) + (2(2) - 1) + (2(3) - 1) + (2(4) - 1) + (2(5) - 1) + \\
(2(6) - 1) + (2(7) - 1) + (2(8) - 1) + (2(9) - 1)
\]

\[= 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 = 81\]

Using these ideas, try to expand and simplify the following summations.

\[\sum_{n=0}^{6} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!}\]

\[= \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720}\]

\[= \frac{720}{720} + \frac{720}{720} + \frac{360}{720} + \frac{120}{720} + \frac{30}{720} + \frac{6}{720} + \frac{1}{720} = \frac{1957}{720}\]

\[\sum_{l=2}^{5} \sqrt{l - 2} = \sqrt{2 - 2} + \sqrt{3 - 2} + \sqrt{4 - 2} + \sqrt{5 - 2}\]

\[= \sqrt{0} + \sqrt{1} + \sqrt{2} + \sqrt{3} \approx 4.15\]

\[\sum_{m=-1}^{4} m(2^m) = -1(2^{-1}) + 0(2^0) + 1(2^1) + 2(2^2) + 3(2^3) + 4(2^4)\]

\[= -1(\frac{1}{2}) + 0 + 1(2) + 2(4) + 3(8) + 4(16)\]

\[= -\frac{1}{2} + 2 + 8 + 24 + 64\]

\[= 97.5\]
Now rewrite the sums below in sigma notation:

\[-2 + (-1 + 0 + 1) + 2 + 3 + 4 + 5 + 6\]

\[= \sum_{n=-2}^{6} n\]

\[2 + 4 + 6 + 8 + 10 + 12\]

\[= \sum_{k=1}^{6} 2k\]

\[\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36}\]

\[= \sum_{m=1}^{6} \frac{1}{m^2}\]

Statistics applies sigma notation to define many of the terms, such as mean, variance, and standard deviation, etc. When we are trying to sum data, we use the following notation: \(a_i\) represents the \(i^{th}\) term of the data and

\[\sum_{i=1}^{n} a_i\]

represents the sum of all \(n\) terms of our data.

Now consider students’ scores from an exam: 85, 74, 99, 61, 93, 81, 78, and 60.

How would we find the mean exam score?

We could add up the test scores and divide by the total number of test scores (8).

How could we use sigma notation to represent the mean exam score?

\[\sum_{i=1}^{8} a_i\]
Suppose you were given \( m \) exam scores, how could you use sigma notation to represent the mean of these exam scores?

\[
\sum_{i=1}^{m} a_i
\]

Another useful application of sigma notation is determining areas of regions. Consider the following region below:

If we know the areas of each of the six region to be \( A_i \), how could we represent the total area of the figure using sigma notation?

\[
\sum_{i=1}^{6} A_i
\]