Application of Derivatives: Profit, Cost, and Revenue

1) How can we find the profit function in terms of the revenue function and cost function?

   The profit function is the revenue function minus the cost function, or
   \[ P(q) = R(q) - C(q) \]

Consider the graphical representations of the cost and revenue functions below:

2) If the cost curve lies above the revenue curve within an interval, what does this tell us about the profit within that interval?

   Along such an interval, cost is greater than revenue. This means that profit will be negative within this interval.

3) If the cost curve lies below the revenue curve within an interval, what does this tell us about the profit within that interval?

   Along such an interval, revenue is greater than cost. This means that profit will be positive within this interval.
Using the graphs of the cost function $C(q)$ and revenue function $R(q)$ shown above, answer the following:

4) Determine the interval(s) where the profit function will be positive.

\[(25, 68)\]

5) Determine the interval(s) where the profit function will be negative.

\[(0, 25) \text{ and } (68, 130)\]

6) How can we determine when the profit function is at its maximum?
   Consider intervals where profit is positive. For every $q$-value, find the difference in $R(q)$ and $C(q)$. The $q$-value with the largest difference is the maximum.

7) Determine where the profit function will be at its maximum.

At about $q=50$, we will have our maximum profit. \((R(50) - C(50) \approx \$433)\)
Recall that marginal cost is the derivative of the cost function and that marginal revenue is the
derivative of the revenue function. Also, recall that the derivative of the function at a point is the
slope of the tangent line of the original function at that point. With this in mind, answer the
following using the graph above:

8) Determine the interval(s) where marginal cost is greater than marginal revenue.
   (This means, when are the slopes of the tangent lines along
   \( C'(q) \) greater than the slopes of the tangent lines along
   \( R'(q) \)?) \((50, 130)\)

9) What does this tell us about our profit function within these interval(s)?
   Since cost is increasing faster than revenue along
   this interval, profit is decreasing.

10) Determine the interval(s) where marginal cost is less than marginal revenue.
   (This means, when are the slopes of the tangent lines along
   \( C'(q) \) greater than the slopes of the tangent lines along
   \( R'(q) \)?) \((0, 50)\)

11) What does this tell us about our profit function within these interval(s)?
   Since revenue is increasing faster than cost along
   this interval, profit is increasing.

12) Determine where the marginal cost is equal to marginal revenue. What does this mean
    about our marginal profit at this \(q\)-value?
    At \( q = 50 \), marginal cost is equal to marginal
    revenue \( (C'(q) = R'(q)) \). This means that our
    marginal profit at this point is 0.

13) Where have you seen this \(q\)-value before? What does it represent?
    In \#7, we said that at \( q = 50 \) is where
    we have our maximum profit.

14) What would \( P'(q) \) be equal to at this \(q\)-value? What is this \(q\)-value also known as?
    Since \( C'(50) = R'(50) \), and \( P'(q) = R'(q) - C'(q) \),
    \( P'(50) = R'(50) - C'(50) = 0 \). We have a
    critical value at \( q = 50 \).
The maximum (or minimum) profit can occur where

Marginal Profit = 0 (i.e. \( P'(q) = 0 \)),

that is, where

Marginal Revenue = Marginal Cost (i.e. \( R'(q) = C'(q) \)).

Recall that marginal cost, marginal revenue, and marginal profit represent the rate of change, or slope, of cost, revenue, and profit, respectively.

Another way to think of these terms is:

- **Marginal cost** at a point represents the increase/decrease in cost to produce one additional item.
- **Marginal revenue** at a point represents the increase/decrease in revenue for selling one additional item.
- **Marginal profit** at a point represents the increase/decrease in profit for producing and selling one additional item.

Examples

1) Revenue is given by \( R(q) = 325q \) and cost is given by \( C(q) = 5000 + 2q^2 \). At what quantity is profit maximized? What is the total profit at this production level?

\[
P(q) = R(q) - C(q) = 325q - (5000 + 2q^2) = -2q^2 + 325q - 5000
\]

\[
P'(q) = -4q + 325. \quad \text{Max occurs when } P'(q) = 0 = -4q + 325.
\]

\[\hat{q} = 81.25, \quad P(81.25) = 8203.125\]

2) Let \( C(q) \) represent the cost, \( R(q) \) represent the revenue, and \( P(q) \) represent the total profit, in dollars, of producing \( q \) items.

a. If \( C'(50) = 75 \) and \( R'(50) = 84 \), approximately how much profit is earned by the 51st item?

\[
P'(50) = R'(50) - C'(50) = 84 - 75 = 9
\]

By making and selling the 51st item, the company will profit an additional \( \$9 \).

b. If \( C'(90) = 75 \) and \( R'(90) = 84 \), approximately how much profit is earned by the 91st item?

\[
P'(90) = R'(90) - C'(90) = 84 - 75 = 9
\]

By making and selling the 91st item, the company will profit an additional \( \$9 \).

c. If \( P(q) \) is a maximum when \( q = 78 \), how do you think \( C'(78) \) and \( R'(78) \) compare? Explain your answer.

When \( P(q) \) is a maximum, \( P'(q) = 0 = R'(q) - C'(q) \).

So \( 0 = R'(78) - C'(78) \).