NONSTANDARD METHODS AND THE ERDŐS-TURÁN CONJECTURE

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1. Introduction

A major open question in combinatorial number theory is the Erdős-Turán conjecture which states that if $A =< a_n >$ is a sequence of natural numbers with the property that $\sum_{n=1}^{\infty} 1/a_n$ diverges then A contains arbitrarily long arithmetic progressions [1]. The difficulty of this problem is underscored by the fact that a positive answer would generalize Szemerédi's theorem which says that if a sequence $A \subset \mathbb{N}$ has positive upper Banach Density then A contains arbitrarily long arithmetic progressions. Szemerédi's theorem itself has been the object of intense interest since first conjectured, also by Erdős and Turán, in 1936. First proved by Szemerédi in 1974 [9], the theorem has been re-proved using completely different approaches by Furstenberg in 1977 [2, 3] and Gowers in 1999 [4], with each proof introducing powerful new methods.

The Erdős-Turán conjecture immediately implies that the primes contain arbitrarily long arithmetic progressions, and it was thought by many that a successful proof for the primes would be the result of either a proof of the conjecture itself or significant progress toward the conjecture. However, very recently Green and Tao were able to solve the question for the primes without generalizing Szemerédi's result in terms of providing weaker density conditions on a sequence guaranteeing that it contain arithmetic progressions.

In this paper we outline some possible ways in which nonstandard methods might be able to provide new approaches to attacking the Erdős-Turán conjecture, or at least other questions about the existence of arithmetic progressions. Heavy reference will be made to results in [8] and [7], and the proofs for all results quoted but not proved here appear in those two sources.

2. Near Arithmetic Progressions

We begin with some definitions that first appear in [8].

Definition 1. *Let* $A \subset \mathbb{N}$*, and let* $I = [a, b]$ *be an interval in* $\mathbb{N}$*. We will write* $\mathbf{l(I)}$ *for the length of* I *(i.e.* $l(I) = b - a + 1$*) and we will write* $\delta(\mathbf{A}, \mathbf{I})$ *or* $\delta(\mathbf{A}, [\mathbf{a}, \mathbf{b}])$ *for the density of the set* A *on the interval* I*. Thus* $\delta(A, I) = \frac{|A \cap I|}{l(I)}$.

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Definition 2. *Let* t, d *and* w *be in* $\mathbb{N}$, *and let* $\alpha \in \mathbb{R}$ *with* $0 < \alpha < 1$. *For* $A \subset \mathbb{N}$ *and* I *an interval in* $\mathbb{N}$ *of length* $l(I)$ *we say that* A *contains a* t*-termed* α*-homogeneous cell of distance* d *and width* w *in* I *or simply a* $<\mathbf{t}, \alpha, \mathbf{d}, \mathbf{w}>$ *cell in* I *iff there exists* $b \in I$ *with* $b + (t-1)d + w$ *also in* I *such that for each* $\nu, \xi = 0, 1, 2, ..., t-1$:

$$\delta(A, [b + \xi \cdot d, b + \xi \cdot d + w]) \geq (1-\alpha)\delta(A, [b + \nu \cdot d, b + \nu \cdot d + w]) \geq (1-\alpha)^2 \delta(A, I).$$

If each $\delta(A, [b + \xi \cdot d, b + \xi \cdot d + w])$ *is simply nonzero, i.e. the intervals are nonempty, then we say that* A *contains a* $<\mathbf{t}, \mathbf{d}, \mathbf{w}>$ *cell.*

For $\beta > 0$ *and* $0 \leq u \leq w$ *we will say that a* $< t, \alpha, d, w >$ *cell is* $\mathbf{u}, \beta$ ***uniform*** *if for each* $\nu = 0, 1, 2, ..., t-1$, *and all* x *such that* $u \leq x \leq w$:

$$(1-\beta)\delta(A, J_\nu) \leq \delta(A, [b + \nu \cdot d, b + \nu \cdot d + x]) \leq (1+\beta)\delta(A, J_\nu),$$

where J_ν *denotes the interval* $[b + \nu \cdot d, b + \nu \cdot d + w]$.

It is clear that an actual arithmetic progression of length t and distance d is an example of a $< t, \alpha, w, d >$ cell with $w = 0$ and α any non-negative number. Furthermore, this cell is u, β uniform for $u = 0$ and any non-negative β. We could view the existence of a $< t, \alpha, d, w >$ cell inside a sequence A as a weak form of an actual arithmetic progression inside A. These cells are "near" arithmetic progressions in some (perhaps rather weak) sense, and intuitively are "nearer" to arithmetic progressions as the size of w decreases. In some of the results that we look at w will be "small" in the sense that the ratio of w to d will be small compared to the ratio of d to the length of the interval I. In other results w will be "small" by actually being bounded by a finite number while d gets arbitrarily large.

Definition 3. *Let* I *be an interval in* $\mathbb{N}$, *and* $A \subset I$, *with* $r > 1 \in \mathbb{R}$ *and* $m \in \mathbb{N}$. *We say that* A *has the* $\mathbf{m}, \mathbf{r}$ ***density property*** *on* I *iff for any interval* $J \subset I$, *if* $l(J) \geq \frac{l(I)}{m}$, *then* $\delta(A, J) \leq r\delta(A, I)$.

Theorem 1 below gives a condition for the existence of "near" arithmetic progressions for any sequence on any interval I in which the density does not drastically increase as the size of the subinterval decreases. More specifically, it provides an absolute constant such that whenever the density of a sequence does not increase beyond a fixed ratio for any subinterval of size greater than the length of I divided by that fixed constant, then the sequence will contain a $< t, \alpha, w, d >$ cell with some relative "smallness" conditions for w.

A complete proof of this theorem appears in [8], but we will outline the proof here, as it provides the clearest illustration of how the use of the nonstandard model provides us with a new set of tools for questions of this type.

Theorem 1. *Let* $h(x)$ *be any increasing real valued function such that* $h(x) > 0$ *whenever* $x > 0$, *and let* $g(x)$ *be any real valued function which approaches infinity as* x *approaches infinity. For all real* $\alpha > 0$, $r > 1$ *and* $j, t \in \mathbb{N}$ *there exists a standard natural number* m *such that for all* $n > m$, *whenever* I *is an interval of length* n *and any nonempty set* $A \subset I$ *has the* m, r *density property on* I *then* A *contains a* u, β *uniform* $< t, \alpha, d, w >$ *cell with* $\frac{u}{w} < h(\frac{w}{d})$, $\frac{w}{d} < h(\frac{d}{n})$, $\beta < h(\frac{d}{n})$ *and* $\frac{n}{g(m)} < d < \frac{n}{j}$. *Furthermore, we may take* w *and* d *to be powers of 2.*

Proof. (Sketch only). Suppose $h(x), g(x), \alpha, j, r, t$ are given as in the statement above and that no such m exists. By "overspill" there exists an M, N in ${}^*\mathbb{N} - \mathbb{N}$ with $M < N$ and a hyperfinite internal set A such that A has the M, r density property on an interval of length N but A contains no $< t, \alpha, d, w >$ cell on this interval with the required properties. Since the conditions are translation invariant we may assume that the interval is $[0, N-1]$. We now define a standard function $f : [0,1] \longrightarrow [0,1]$ by:

$$f(x) = st\left(\frac{|A \cap [0, xN]|}{|A \cap [0, N]|}\right).$$

Using the fact that A has the M, r density property on $[0, N]$ it is not difficult to show that $f(x)$ satisfies a Lipschitz condition with constant r. Thus, the function f is absolutely continuous, differentiable almost everywhere and equal to the integral of its derivative. Since $f(1) = 1$, $f(0) = 0$ and f is the integral of its derivative, it must be that the lebesgue measure of $\left\{x : f'(x) \geq \left(1 - \frac{\alpha}{4}\right)\right\}$ is nonzero. Thus, there exists a real number $c \geq 1$ such that the lebesgue measure of the set

$$E = \left\{x : c - \frac{\alpha}{4} \leq f'(x) \leq c\right\}$$

is nonzero.

By using the lebesgue density theorem it is straightforward to show that any set of positive measure contains arbitrarily long arithmetic progressions, and that, in fact, these progressions may have arbitrarily small differences between elements. This allows us to obtain a $< t, \alpha, D, W >$ cell, with $D, W \in {}^*\mathbb{N} - \mathbb{N}$ with the property that there exists $B \in {}^*\mathbb{N} - \mathbb{N}$ such that

$$st\left(\frac{B}{N}\right), st\left(\frac{B+D}{N}\right), st\left(\frac{B+2D}{N}\right), ..., st\left(\frac{B+(t-1)D}{N}\right)$$

forms an arithmetic progression in E. The α homogeneity follows from the definition of E. The fact that f is differentiable at each point in E allows us to obtain the uniformity condition, and allows us to take U, D and W arbitrarily small but not infinitesimal to N. This, in turn, allows us the freedom to make those quantities powers of 2.

We are thus able to obtain a U, β uniform $< t, \alpha, D, W >$ cell for A in $[0, N-1]$ with all the properties required in the theorem, contradicting our assumption. □

Definition 4. *For A a sequence of positive integers we define the* ***upper Banach Density*** *of A or* $\mathbf{BD(A)}$ *by:*

$$BD(A) = \inf_{x \in \mathbb{N} - \{0\}} \max_{a \in \mathbb{N}} \frac{|A \cap [a+1, a+x]|}{x}.$$

Upper Banach density is often simply called Banach density, and is sometimes referred to in the literature as strong upper density, with notation $d^*(A)$ in place of $BD(A)$. That notation is used in [7].

The theorem allows us to obtain some results about the existence of uniform $< t, \alpha, d, w >$ cells in sequences that are relatively sparse (certainly too sparse to necessarily contain actual arithmetic progressions). The theorem below, also proved in [8], is of this type.

Theorem 2. *Let $\alpha > 0$, and $t > 2 \in \mathbb{N}$ be given, $h(x)$ be any continuous real valued function such that $h(x) > 0$ whenever $x > 0$ and let A be a sequence in $\mathbb{N}$ with the property that for all $\varepsilon > 0$, $|A \cap [0, n-1]| > n^{1-\varepsilon}$ for sufficiently large n. Then for sufficiently large n, A contains a u, β uniform $< t, \alpha, d, w >$ cell on $[0, n-1]$ with w and d powers of 2 and such that:*

$$\frac{u}{w} < h(\frac{w}{d}), \quad \frac{w}{d} < h\left(\frac{\log d}{\log n}\right), \text{ and } \beta < h\left(\frac{\log d}{\log n}\right).$$

The condition that $\frac{w}{d} < h\left(\frac{\log d}{\log n}\right)$ is not very strong, and it would be desirable to improve this weak "smallness" condition. The theorem below shows that, even for $t = 3$, if we keep the density condition on A as above then we cannot improve this smallness condition to $\frac{w}{d} < \left(\frac{d}{n}\right)^{\alpha}$ for any $\alpha > 0$, even if we do not insist on any homogeneity or uniformity conditions.

Theorem 3. *Let $\alpha > 0$. There exist constants $r > 0$ such that for arbitrarily large n there are subsets A of $[0, n-1]$ such that*

$$|A \cap [0, n-1]| > \frac{n}{2^{r \log\log n \sqrt{\log n}}},$$

and yet A contains no $< 3, d, w >$ cell in $[0, n-1]$ satisfying $\frac{w}{d} < \left(\frac{d}{n}\right)^{\alpha}$, with w and d a power of 2.

Here we recall that a $< t, d, w >$ cell is merely a collection of t intervals in arithmetic progression on which A is nonempty.

Proof. Since the statement is strictly stronger as α decreases, we will assume that $\alpha \leq 1$. In [5] (p. 98) it is shown that there exists a constant $c > 0$ and a sequence A satisfying

$$|A \cap [0, n-1]| > \frac{n}{e^{c\sqrt{\log n}}} \text{ for sufficiently large } n$$

that contains no 3-term arithmetic progression. This result is due to Behrend. For convenience we will adjust the constant and use log base 2 here, and also replace e with 2. By adjusting the constant if necessary (and using $2A$) we may assume that A contains no two consecutive numbers. We may also translate so that $0 \in A$ without changing the density condition. Thus, we begin with a sequence A which contains 0 and no two consecutive numbers and satisfies

$$|A \cap [0, n-1]| > \frac{n}{2^{c\sqrt{\log n}}} \text{ for sufficiently large } n.$$

Let $N \in {}^*\mathbb{N} - \mathbb{N}$ and let

$$\beta = (1/2)^{2/\alpha}; \quad m_0 = N; \quad m_1 = \text{ the largest power of 2 less than } \beta^{(1+\alpha/2)}N$$

$$m_{k+1} = \text{ the smallest power of 2 greater than } \left(\frac{m_k}{N}\right)^{\alpha} m_k$$

$$L = \text{ the smallest number such that } m_{L+1} \leq 1.$$

We now wish to show by induction that

$$m_k \leq \beta^{(1+\alpha/2)^k} N. \tag{2.1}$$

To see this we note that for $k = 1$ the definition of m_1 guarantees this. By the construction we have

$$\left(\frac{m_k}{N}\right)^{\alpha} m_k < m_{k+1} \leq 2\left(\frac{m_k}{N}\right)^{\alpha} m_k.$$

so that, assuming the induction hypothesis,

$$\begin{aligned} m_{k+1} &\leq 2\left(\frac{m_k}{N}\right)^{\alpha} m_k \leq 2\left(\beta^{(1+\alpha/2)^k}\right)^{\alpha} \beta^{(1+\alpha/2)^k} N = 2\left(\beta^{\alpha(1+\alpha/2)^k}\right) \beta^{(1+\alpha/2)^k} N \\ &= 2\left(\beta^{\alpha/2(1+\alpha/2)^k}\right)\left(\beta^{\alpha/2(1+\alpha/2)^k}\right) \beta^{(1+\alpha/2)^k} N \leq 2\beta^{\alpha/2}\beta^{\alpha/2(1+\alpha/2)^k+(1+\alpha/2)^k} N \\ &\leq \beta^{\alpha/2(1+\alpha/2)^k+(1+\alpha/2)^k} N = \beta^{(1+\alpha/2)^{k+1}} N, \end{aligned}$$

completing the induction step, and establishing 2.1 above.

We will define a subset B of $[0, N-1]$ with the property that it contains no $< 3, d, w >$ cell in $[0, N-1]$ satisfying $\frac{w}{d} < \left(\frac{d}{N}\right)^{\alpha}$, with w and d a power of 2, and such that

$$|B \cap [0, N-1]| > \frac{N}{2^{cL\sqrt{\log N}}}$$

by essentially using block copies of initial segments of ${}^*A$. More specifically we let

$$B_1 = [0, m_1 - 1]$$

and, for $1 \leq k < L$

$$i \in B_{k+1} \text{ iff } \left\lfloor \frac{i'}{m_{k+1}} \right\rfloor \in {}^*A, \text{ where } i' \text{ is the remainder of } i \bmod m_k.$$

When k is finite, the fact that $0 \in {}^*A$ means that B_k intersected with $B_1 \cap \ldots \cap B_{k-1}$ has cardinality at least a noninifinitesimal multiple of that of $B_1 \cap \ldots \cap B_{k-1}$. When k is in ${}^*\mathbb{N} - \mathbb{N}$ the density condition on A guarantees that at least a $\frac{1}{2^{c\sqrt{\log n}}}$ portion of B_k intersects with $B_1 \cap \ldots \cap B_{k-1}$, where $n < N$. Thus

$$B_1 \cap \ldots \cap B_k \text{ has cardinality at least } \frac{N}{2^{ck\sqrt{\log N}}}. \tag{2.2}$$

We now let

$$B = B_1 \cap \ldots \cap B_L.$$

Now suppose that B contains a $< 3, d, w >$ cell on $[0, N-1]$ with $\frac{w}{d} < \left(\frac{d}{N}\right)^{\alpha}$ where both w and d are powers of 2. We show that this forces an actual arithmetic progression of length 3 in ${}^*A$ and thus in A (by transfer), contradicting our assumption about A.

To see this, we let i be such that $m_{i+1} \leq d < m_i$. Then since ${}^*A$ contains no two consecutive numbers, the $< 3, d, w >$ cell on $[0, N-1]$ must be completely contained inside one of the blocks of length m_i, i.e. inside some $[\nu m_i, (\nu+1)m_i]$. But

$$w \leq \left(\frac{d}{N}\right)^{\alpha} d < \left(\frac{m_i}{N}\right)^{\alpha} m_i \leq m_{i+1},$$

and since w, d and the m_k's are powers of 2

$$w | m_{i+1} \text{ and } m_i | d,$$

so that there exist 3 intervals of length m_{i+1} inside $[\nu m_i, (\nu+1)m_i]$ which contain elements of B and are in arithmetic progression. By the construction this means that ${}^*A$ contains an arithmetic progression of length 3, and then by transfer, so does A.

It remains to estimate L in terms of N. From 2.1 we see that

$$m_k \leq 1 \text{ when } \beta^{(1+\alpha/2)^k} N \leq 1 \text{ i.e. } N \leq (1/\beta)^{(1+\alpha/2)^k}$$

so that

$$\log N \leq (1+\alpha/2)^k \log(1/\beta)$$

or

$$\log\log N \leq k\log(1+\alpha/2) + \log\log(1/\beta).$$

Thus

$$m_k \leq 1 \text{ whenever } k \geq \frac{\log\log N - \log\log(1/\beta)}{\log(1+\alpha/2)}.$$

This and the definition of L now yield

$$L+1 \leq \frac{\log\log N - \log\log(1/\beta)}{\log(1+\alpha/2)}.$$

The above inequality, the definition of B and 2.2 imply that for any $r > c/\log(1+\alpha/2)$

$$|B \cap [0, N-1]| > \frac{N}{2^{r \log\log N \sqrt{\log N}}}.$$

For such an r the result now follows by transfer. $\square$

We note that the density condition given in the theorem above is stronger than simply being greater than $n^{1-\varepsilon}$ for sufficiently large n, since $n^{1-\varepsilon}$ is of the form

$$\frac{n}{2^{c\log N}} < \frac{n}{2^{\log\log n \sqrt{\log n}}} \text{ for large } n.$$

Thus, weak as theorem 2 is in its smallness conditions, there are clear limits to how much it can be strengthened for sets of this relative sparseness.

It appears to be more promising to look at denser sequences in the hope of maintaining a stronger "smallness" condition. The theorem below is proved in [8], and provides just one possible example of conditions like this that might provide a means for approaching deep questions about arithmetic progressions. The proof of theorem 4 below is similar to that of the proof of theorem 3 given above. In particular these proofs illustrate how "smallness" conditions that may not seem very strong can be used to show that somewhat denser sets contain actual arithmetic progressions.

Theorem 4. *The Erdős-Turán conjecture follows if we can show that for fixed t and constant $c > 0$, there exists n_0 such that for all $n > n_0$, whenever the sequence A satisfies*

$$|A \cap [0, n-1]| > \frac{n}{(c\log n)^{2\log\log n}}$$

then A contains a $< t, d, w >$ cell on $[0, n-1]$ with $\frac{w}{d} < \frac{d}{n}$ where both w and d are powers of 2.

3. The Interval-Measure Property

The conditions given below are natural from the nonstandard perspective and not at all so from the standard perspective. They might provide another means of attacking questions about arithmetic progressions. These definitions first appear in [7].

Definition 5. *Let A be an internal subset of ${}^*\mathbb{N}$, $y, z \in {}^*\mathbb{N}$ with $z - y \in {}^*\mathbb{N} - \mathbb{N}$. Then we say that A has the* ***IM (interval-measure)*** *property on $[y, z]$ iff for every real, standard $\beta > 0$ there is a real, standard $\alpha > 0$ such that whenever $[u, v] \subset [y, z]$ with $v - u \in {}^*\mathbb{N} - \mathbb{N}$ and the largest gap of A on $[u, v]$ is $\leq \alpha(v - u)$ then*

$$\lambda\left(st\left\{\left(\frac{a-y}{z-y}\right) : a \in A \cap [y, z]\right\}\right) \geq 1 - \beta.$$

If A is a standard subset of $\mathbb{N}$, we say that A has the ***SIM (standard interval-measure)*** *property iff ${}^*A$ has the IM property on every interval $[y, z] \subset {}^*\mathbb{N}$ with $z - y \in {}^*\mathbb{N} - \mathbb{N}$, and*

$$\lambda\left(st\left\{\left(\frac{a-y}{z-y}\right) : a \in {}^*A \cap [y, z]\right\}\right) > 0 \text{ on some such interval } [y, z].$$

Here λ is used to denote lebesgue measure.

A somewhat cumbersome but standard equivalent definition for the SIM property is given in [7]. Through the use of the standard equivalent it is easy to see that If A has the SIM property then given $\beta > 0$ there exists a fixed $\alpha > 0$ that works for every infinite interval. The theorem below follows immediately from theorem 3.2 in that same work.

Theorem 5. *Let $t \in \mathbb{N}$, $0 < \beta < 1/t$, and suppose that $A \subset \mathbb{N}$ has the SIM property, with α corresponding to the given β. Then there exists a constant $j \in \mathbb{N}$ such that whenever A contains a $< \mathbf{t}, \mathbf{d}, \mathbf{w} >$ cell consisting of the intervals $[b+\xi \cdot d, b+\xi \cdot d+w]$ for $0 \leq \xi \leq t-1$ in which the largest gap of A on each of these intervals is $\leq \alpha w$ then A contains $< \mathbf{t}, \mathbf{d}, \mathbf{j} >$ subcell, i.e. consisting of intervals of the form $[b' + \xi \cdot d, b' + \xi \cdot d + j] \subset [b + \xi \cdot d, b + \xi \cdot d + w]$.*

This result is significant in that a fixed constant size to the intervals is a much stronger "smallness" condition than was achieved in previous results. However, the assumption that A is an SIM set is a strong condition. It is shown in [7] that any sequence $A = \langle a_n \rangle$ in which $\lim_{n\to\infty}(a_{n+1} - a_n) = \infty$ does not have the SIM property, so no pure density condition weaker than positive banach density can imply that A contains an SIM set. On the other hand, sets may certainly have the SIM property without having positive Banach density. Even the question of whether or not positive Banach density is sufficient for a sequence to contain an SIM set is still open. A positive solution to either of the conjectures below would be a major step toward establishing the SIM condition as a useful tool for questions of this type. Since the two conjectures together imply Szemerédi's theorem, at least one of them is certain to be quite difficult.

Conjecture 1. *Let $A \subset \mathbb{N}$ have positive Banach density. Then A contains a subset B with the SIM property.*

Conjecture 2. *Every set $A \subset \mathbb{N}$ with the SIM property contains arbitrarily long arithmetic progressions.*

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