Some questions and answers about “fixed point traps” in the plane

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The results and questions are motivated by efforts to use nonstandard methods to prove results related to the Plane Fixed Point Problem, which asks if every non-separating plane continuum has the fixed point property. A continuum is a compact, connected set, and non-separating means that its complement is also connected. The question dates to the time of Brouwer’s proof that closed disks have the fixed point property. In an overview of continuum theory Ingram remarks that Ayres seems to be the first to ask this question in print in 1930.

So, we are interested in continuous functions from some non-separating plane continuum to itself, and are looking for techniques that help us find fixed points for such mappings. More specifically we are trying to use nonstandard models to create more situations where a classic “dog chases rabbit” (terminology used by Hagopian and others) argument works. The idea of this type of argument is to move from points toward their images in such a way that the images are “trapped” and chased down by the points until a fixed point is found.
We will call an internal subset of the nonstandard plane \( IL\text{-chainable} \) (for “infinitesimally linearly chainable”) if it can be covered by a hyperfinite collection of open sets \( G_0, \ldots, G_n \) of infinitesimal diameter with the “linear chain” condition that \( G_i \cap G_j \) is nonempty iff \( |i - j| \leq 1 \).

A standard set that can be covered by a finite chain of sets \( G_i \) as in the above definition is called \( \delta\text{-chainable} \) if each of the chaining sets are of diameter less than \( \delta \). A set is called \textit{chainable} if it is \( \delta\text{-chainable} \) for all \( \delta > 0 \). So, \( IL\text{-chainable} \) sets are just those that satisfy the usual definition of being \( \delta\text{-chainable} \) inside the nonstandard model for some infinitesimal \( \delta > 0 \).
δ chainable sets form “traps” that allow us to conclude that points are mapped within δ of themselves, and thus an IL chainable set forms a fixed point trap for standard functions under similar conditions. Even then the condition that allows “entrance” or “exit” to the trap must be at an end of the chain without additional conditions.

not a fixed point trap

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If $a_1$ and $a_2$ are two points in the plane or the nonstandard plane we will write $L[a_1, a_2]$ for the straight line segment from $a_1$ to $a_2$. If $A$ is an arc (either standard or internal) and $a_1$ and $a_2$ are two points on $A$ then we will write $A[a_1, a_2]$ for the subarc from $a_1$ to $a_2$. If $S$ is a simple closed curve in the plane or the nonstandard plane we will write $V_S$ for the bounded region whose boundary is $S$ ($V_S$ exists and is well-defined by the Jordan Curve Theorem).
Nonstandard methods are useful in obtaining “thin” sets that can act as fixed point traps.

Sets that contain no standard point and have little intersection with the nonstandard counterpart of a standard set are particularly important.
**Definition**

We will say that a set \( A \subseteq \mathbb{R}^2 \) contains a **Y-set** if there exist four points \( a, b, c, \) and \( x \) in \( A \), arcs \( C_{ax}, C_{bx}, \) and \( C_{cx} \) in \( A \) intersecting only at \( x \), from \( a \) to \( x \), \( b \) to \( x \), and \( c \) to \( x \), respectively, such that none of the points \( a, b, \) or \( c \) are infinitesimally close to any point on the arcs joining the others to \( x \) (thus, for example, no point of \( C_{ax} \) is infinitesimally close to \( b \) or to \( c \)).

If \( \delta > 0 \) we will say that a set \( A \) in the plane contains a **size \( \delta \) Y-set** if there exist four points \( a, b, c, \) and \( x \) in \( A \), and arcs \( C_{ax}, C_{bx}, \) and \( C_{cx} \) in \( A \) intersecting only at \( x \) from \( a \) to \( x \), \( b \) to \( x \), and \( c \) to \( x \), respectively, such that none of the points \( a, b, \) or \( c \) are within \( \delta \) of any point on the arcs joining the others to \( x \) (thus, for example, no point of \( C_{ax} \) is within distance \( \delta \) of \( b \) or \( c \)).
Theorem

Let $V \subset ^*\mathbb{R}^2$ be a bounded (internal) region (i.e. contained in some $^*B(0, r)$ for a standard real $r > 0$) bounded by a simple closed curve $S$ and suppose that $\overline{V}$ contains no standard points, $st(V)$ does not disconnect the plane, and that there exists an arc $A$ of infinitesimal length on $S$ such that $^*st(V) \cap S \subset A$. Then $V$ contains no $Y$-set.
If we remove the condition that $V$ contains no standard points there are simple counterexamples. For example, we may let the blue line segments below form a standard set and the green border be $S$, where every point of $S$ stays within an infinitesimal of the blue set. The blue set itself is clearly a $Y$-set, and the other conditions of the theorem are satisfied.
If we move the blue set over and up by an infinitesimal amount, we cannot surround it by a border that intersects $st(V)$ only on one small arc.
Containing no nonstandard points is somewhat subtle. If an internal arc in the complement of a continuum only intersects a standard arc in two points and no standard point is on the closed shared part of the arc, then the region can contain no standard points. However, the following example is interesting.
The following standard result follows easily from the theorem above:

Let $E$ be a non-separating compact set in the plane, and $R_n$ be a sequence of regions bounded by simple closed curves $S_n$ with the property that for any $k$ and any finite collection of points in the plane $\{p_1, p_2, p_3, \ldots, p_k\}$ there exists an $n$ such that $\{p_1, p_2, p_3, \ldots, p_k\} \cap \overline{R_n} = \emptyset$ and $E \cap S_n \subset A_n$, where $A_n$ is an arc of length less than $1/k$ and all points of $S_n$ are within $1/k$ of some point in $E$. Then for all $\delta > 0$ there exists an $n$ such that $R_n$ contains no size $\delta$ $Y$ set.
A *simple triod* is a union of three arcs that meet at a single point. A classic result of Moore’s asserts that it is not possible to embed uncountably many disjoint triods in the plane. It is not difficult to show that the result above yields a version of this even if we only use the case in which each boundary above does not intersect $E$ at all (rather than on a small arc). This version has an extraneous condition that $E$ not separate the plane. The corollary is:

For every $\delta > 0$ no non-separating compact set in the plane can contain infinitely many disjoint connected components each of which contains a size $\delta$ Y-set.
Containing no Y-set is not nearly as strong as being IL-chainable. Here is a set that is not IL-chainable but contains no Y-set:
This same example also shows that the conditions of the theorem imply more than simply that the region contains no Y set, since the example above cannot be placed in a region in the nonstandard plane in such a way that the conditions of the theorem hold. The standard part of the horizontal segment must intersect any containing region in more than a small arc.

It seems possible that these conditions imply that the region has infinitesimal *symmetric span*. I can show this assuming that there is no intersection at all with \( *E \) from a paper by Oversteegen and Tymchatyn. Such sets would have the fixed point property for a standard function from the set to itself, but it is not clear what additional conditions would allow us to use these as fixed point traps if it is simply a piece of a larger set.
If we go back to the example from the previous picture and imagine it as the outside border of a thin region we see an intriguing example of a region that acts as a fixed point trap for standard mappings but might have internally continuous functions that do not come close to trapping fixed points. The next few diagrams illustrate this.
a fixed point trap pt 1
a fixed point trap pt 2
a fixed point trap pt 3
In addition to finding regions that form fixed point traps, we need to create situations in which points get mapped inside in an appropriate way. Every non-separating plane continuum is the intersection of topological disks. The standard part can always be covered by an internal collection of infinitesimal disks whose union is a topological disk. It appears that a crucial question is to find out under what conditions a region may be covered with infinitesimal disks in such a way that the boundaries never form connected regions in the complement with standard points on the boundary or in the interior.
I don’t have examples in which it is certain that this can’t be done, although it seems likely they exist. Continua which don’t have the fixed point property are known to be complicated in some ways. For example there must exist a simple dense canal whose limit points define the boundary. But there are unknown cases that are fairly simple in other ways. It is not even known for sets which are close to chainable in the sense that they are intersections of collections of disks that have only a fixed finite number of junction points. It seems very likely that in that case we can always cover in such a way that the situation in the slide above does not occur, but I have not been able to show that yet.
Questions.

Under what conditions can regions with no Y sets become fixed point traps?

Under what condition can regions with the conditions given in the main theorem become fixed point traps?

Which continua have the property that they can covered in such a way that the boundaries never form connected regions in the complement with standard points on the boundary or in the interior?