An example of the use of nonstandard methods in continuum theory

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Nonstandard analysis, first developed by Abraham Robinson in the early 1960’s [6], makes use of the fact that there are other *models* besides the usual ones that satisfy all the same mathematical statements that can be made in *First Order Logic*. The goal is to exploit what is different in these models - the existence of a wide array of “actualized” limits including “infinitesimals”- and also what is the same - all mathematical properties that can be expressed formally in a very precise sense. Although the first big success of nonstandard methods was Robinson’s proof about nontrivial invariant subspaces for polynomially compact operators in $l^2$, most of the standard results obtained by nonstandard means so far have usually centered around problems that use the “Loeb measure” construction.
Nonstandard methods were used to prove the standard theorem below:

**Definition**

If $\delta > 0$ we will say that a set $A$ in the plane contains a **size $\delta$ Y-set** if there exist four points $a, b, c,$ and $x$ in $A,$ and arcs $C_{ax}, C_{bx},$ and $C_{cx}$ in $A$ intersecting only at $x$ from $a$ to $x,$ $b$ to $x,$ and $c$ to $x,$ respectively, such that none of the points $a, b,$ or $c$ are within $\delta$ of any point on the arcs joining the others to $x$ (thus, for example, no point of $C_{ax}$ is within distance $\delta$ of $b$ or $c$).
Theorem

Let $E$ be a non-separating plane continuum that contains a simple dense canal, and let $C$ be an infinite arc (i.e. the image of a one-to-one continuous function from $[0, \infty)$ to $\mathbb{R}^2$) in the complement of $E$ with the property that $\overline{C} - C = E$ and for every $\varepsilon > 0$ there exists a point $p$ on $C$ such that all points on the arc beyond $p$ are on a transverse cross cut of distance less than $\varepsilon$. Then for any $\delta > 0$ there exist points $p_1$ and $p_2$ on $C$, and an arc of a circle $A$, such that $C[p_1, p_2]$ together with $A[p_1, p_2]$ forms a simple closed curve $S$ that is within $\delta$ of every point in $E$ and is such that $V_S$ contains no size $\delta$ Y-set.
Conjecture: The conclusion of the theorem can be improved to ..... “and is such that $V_S$ is $\delta$-chainable.” (Can be covered by a chain with mesh less than $\delta$).

The goal in relation to the fixed point problem would then be to try to create a scenario in which we use these bounded complementary domains to trap mapped points and create fixed points.
The result is proved in two steps. The first step is to show a nonstandard result that is natural within the nonstandard framework. It roughly translates into a standard result as follows:

**Theorem** Let $E$ be a non-separating plane continuum with no interior, and $R_n$ be a sequence of regions bounded by simple closed curves $S_n$ with the property that for any $k$ and any finite collection of points in the plane $\{p_1, p_2, p_3, \ldots, p_k\}$ there exists an $n$ such that $\{p_1, p_2, p_3, \ldots, p_k\} \cap \overline{R_n} = \emptyset$ and $E \cap S_n \subset A_n$, where $A_n$ is an arc of length less than $1/k$ and all points of $S_n$ are within $1/k$ of some point in $E$. Then for all $\delta > 0$ there exists an $n$ such that $R_n$ contains no size $\delta$ $Y$ set.
The second step is to show that, given a simple dense canal, we can always cut off portions of the complement using arcs of circles in such a way that the hypotheses of the theorem above are satisfied. Thus, we need to prove the following standard result:

For any finite collection of points \( \{x_1, x_2, \ldots, x_k\} \) in \( E \) and any standard \( \delta > 0 \) there exist points \( a \) and \( b \) on \( C \) and an arc of some circle \( A \) such that \( C[a, b] \) is within \( \delta \) of every point of \( E \), \( a \) and \( b \) are within \( \delta \) of each other, and the portion of \( C \) from \( a \) to \( b \) together with \( A[b, a] \) forms a simple closed curve whose interior does not contain any of the points \( \{x_1, x_2, \ldots, x_k\} \).
Here I give a very brief outline of nonstandard methods. There is a more extensive outline and slides from two talks in Sacramento that give more details about the methods and my recent paper. These are available at my web site, along with the paper itself:

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There are nonstandard versions of the reals, the plane, the integers and, in fact, every infinite set. We indicate the nonstandard version of a standard set with a “*”, for example by “*\mathbb{R}^2.” Intuitively, the nonstandard reals, for example “think” they are the reals in the sense identified above - they satisfy all the same mathematical statements within the framework of first order logic. For example, they satisfy the statement that says that between any two reals there is another real number, expressed by:

\[ \forall x \forall y ((x < y) \Rightarrow \exists z ((x < z) \land (z < y))) \]
They even satisfy the statement

\[ \forall x((x \in P(\mathbb{R}) \land (x \neq \emptyset)) \land (\exists y(\forall z \in x(z \leq y))) \Rightarrow \\
\exists u(\forall z \in x(z \leq u)) \land \forall v(\forall z \in x(z \leq v)) \Rightarrow (u \leq v)). \]

This statement says that every nonempty subset of the real numbers that has an upper bound has a least upper bound. Since it is expressible in this way, the nonstandard model must also satisfy this statement. And yet, the nonstandard model contains *infinitesimals* -numbers that are not 0 but whose distance to zero is less than every standard positive number. The set of all infinitesimals has an upper bound (the number .4, for example) but has no least upper bound, which seems to contradict what we just said. However, the set of all infinitesimals is not a set that the nonstandard model “recognizes” as a set.
The *internal* sets are the ones that the nonstandard model recognizes as sets, and it is these sets which must have the same properties as the standard model. When we move to the nonstandard world, the beginning of the statement above corresponds to insisting that $\forall x (x \in *P(\mathbb{R}))........$, and those elements of $*P(\mathbb{R})$ are exactly the internal sets. Basically these sets are definable inside the nonstandard “universe.” Sets such as the infinitesimals are in $P(*\mathbb{R})$ but are not in $*P(\mathbb{R})$.

A more concrete view of what these sets look like: When we define the rationals starting with the reals we might use equivalence classes of Cauchy Sequences of rationals, and we identify two sequences $<a_n>$ and $<b_n>$ as being equivalent iff $\lim_{n \to \infty} (a_n - b_n) = 0$. One way to think of a nonstandard universe is that all the elements, including the sets, are sequences like this, but we have a more refined equivalence relation. We use the concept of a *nonprincipal ultrafilter* on the natural numbers to make this definition.
An ultrafilter is a collection of subsets (of \( \mathbb{N} \) in this case) that is closed under finite intersections and has the property that if \( A \) is in the ultrafilter and \( A \subseteq B \) then \( B \) is in the ultrafilter, and that every set or its complement is in the ultrafilter. A simple example of an ultrafilter is the collection of all sets that contain the number 3.

This example is too simple (it is a \textit{principal} ultrafilter). We will use an ultrafilter that contains no finite sets (a \textit{nonprincipal} ultrafilter). We can use the axiom of choice to show that such ultrafilters exist. We could also think of this as a finitely additive \( \{0, 1\} \) measure (the sets in the ultrafilter have measure 1, those that are not in the ultrafilter have measure 0).
One way to make $^\ast\mathbb{R}$ now is to let the set consist of all infinite sequences of reals, and define two sequences $< a_n >$ and $< b_n >$ to be equivalent iff $\{n : a_n = b_n\}$ is in the ultrafilter (equivalently, has measure 1 in our $\{0, 1\}$ measure). Similarly $< a_n > \leq < b_n >$, for example, if $\{n : a_n \leq b_n\}$ is in the ultrafilter. The real number $a$ is now identified with $< a, a, a, a, ... >$, and we can see that the number $< 1, \frac{1}{2}, \frac{1}{3}, ... >$ is an example of an infinitesimal. This same construction applies to sets as well as elements. The internal sets are precisely the ones that are of the form $< A_n >$ where each $A_n$ is a standard set. Two nonstandard sets are equal iff the set of indices on which they agree is in the ultrafilter. For example: $< 1, \frac{1}{4}, \frac{1}{9}, ... > \in < (0, 1), (0, \frac{1}{2}), (0, \frac{1}{3}), ... >$

The element on the left is an infinitesimal in $^\ast\mathbb{R}$ and the one on the right is an internal subset of $^\ast\mathbb{R}$ (the open interval from 0 to the infinitesimal $< 1, \frac{1}{2}, \frac{1}{3}, ... >$).
We now try to get the best of both worlds. We work in a language that is quite strong, but not strong enough to characterize sets up to isomorphism, so there are these nonstandard models. When convenient, we use the fact that infinitesimal or infinite elements exist, and yet we often use the *transfer* principle that says every definable statement true in the nonstandard world is also true in the standard world.

We usually work inside a part of set theory that contains all the standard objects we are interested in. In this case that would include the plane, all subsets of the plane, and perhaps even all functions from the plane to the plane, or all possible topologies on the plane, etc. We use ultrafilters over larger index sets because this leads to structures that have more *saturation*.
We write \( st(a) \) for the unique real number (if \( a \) is a nonstandard real) or the unique point in the plane (if \( a \) is in the nonstandard plane), that is within an infinitesimal distance of \( a \), assuming that any such number or point exists (there are nonstandard real numbers larger than every standard number, and these have no standard part). If \( A \) is a subset of the nonstandard reals or the nonstandard plane then \( st(A) \) is the set of all standard parts of \( A \).

We write \( st(A) \) for the set of all standard parts of a set \( A \) in the nonstandard universe.
Some simple nonstandard equivalents:

**Proposition**

i) The set $A$ is open iff for all $a \in A$, whenever $b \approx a$ then $b \in ^* A$.

ii) The set $A$ is closed iff whenever $a \in ^* A$ and $st(a)$ exists then $st(a) \in A$.

iii) The set $A$ is compact iff every point of $^* A$ is near some standard point of $A$.

And some simple but useful results:

i) If $A$ is any internal set in the plane then $st(A)$ (the set of all standard points infinitesimally close to points in $A$) is closed.

ii) If $A$ is internal, connected and bounded (i.e. contained in some $^* B(0, r)$ for a standard real $r > 0$) in the nonstandard plane then $st(A)$ is connected.
Some examples:

*$(0, 1)$ is the nonstandard version of the interval $(0, 1)$, so it includes numbers within an infinitesimal of $0$ and $1$. We note that every point near any standard point in $(0, 1)$ is in $*(0, 1)$.

\[ st( *(0, 1)) = [0, 1], \] a closed set.

\[ *st( *(0, 1)) = *[0, 1], \] the nonstandard equivalent of $[0, 1]$.

Every point in $*[0, 1]$ is near some standard point of $[0, 1]$. 
A standard function $f$, from $\mathbb{R}^2$ to $\mathbb{R}^2$, is continuous on a domain $D$ iff for all $a, x \in *D$ where $a$ is standard, $(x \approx a) \Rightarrow (f(x) \approx f(a))$. The function $f$ is uniformly continuous on $D$ iff for all $x, y \in *D$ $(x \approx y) \Rightarrow (f(x) \approx f(y))$.

To clarify the difference between the two definitions, note that a retract from the punctured disk to the boundary, for example, will take two points that are near each other (two points that are both within an infinitesimal of the center) to points that may be on opposite sides of the circle. Thus, continuity alone does not guarantee the stronger condition given in the proposition.
An example of a nonstandard proof of a standard result

**Proposition:** If $E$ is a non-separating plane continuum then for all $\delta > 0$ there exists a set $D$ homeomorphic to the disk such that $E \subset D$ and every point of $D$ is within $\delta$ of a point in $E$ (thus $E$ is a countable intersection of sets homeomorphic to the disk).

**Proof.**

Let $\delta > 0$ be standard and let $\zeta > 0$ be infinitesimal. Let $K$ be an internal finite union of closed $\zeta$-balls that cover $E$, and let $D$ consist of all points in the nonstandard plane “enclosed” by $K$, i.e. all points disconnected from infinity by $K$. We note that $st(K) = E$ since every point of $K$ is within an infinitesimal distance of $E$ and $E$ is compact. There can be no $\delta$-ball contained in $D$ that does not intersect $E$, for if so the center of such a ball would be a point not in $E$ that is disconnected from infinity by $st(K) = E$ and thus $E$ would disconnect the plane. Thus, in the nonstandard universe there exists such a $D$ with the desired properties, and this is true in the standard universe by transfer.
In the picture below, $D$ consists of all points enclosed by the outer boundary. It is not easy to see that small circles cut off small regions. But infinitesimal circles must cut off infinitesimal regions, or we can easily contradict the condition that $E$ does not separate the plane.
Proposition: A set $A$ in the plane is a pseudo-arc iff $^*A$ can be covered by an internal chain $\{C_1, \ldots, C_n\}$ of connected open sets of infinitesimal diameter such that if $C_i$ and $C_j$ are any two links in the chain and $B_i$ and $B_j$ are standard balls centered at $st(C_i)$ and $st(C_j)$ respectively, then the subchain starting at $C_i$ and ending at $C_j$ must include a subchain that intersects $B_j$ and then intersects $B_i$ before it gets to $C_j$ (we note that $st(C_i)$ and $st(C_j)$ are both single points since the diameters are infinitesimal).

This allows us the use of a single chain to characterize these important sets.
Back to Simple Dense Canals
Suppose that $C$ is a simple dense canal for the set $E$, and we want to show the following standard result:

For any finite collection of standard points $\{x_1, x_2, \ldots, x_n\}$ in $E$ and any standard $\delta > 0$ there exist points $a$ and $b$ on $C$ and an arc of some circle $A$ such that $C[a, b]$ is within $\delta$ of every point of $E$, $a$ and $b$ are within $\delta$ of each other, and the portion of $C$ from $a$ to $b$ together with $A[b, a]$ forms a simple closed curve whose interior does not contain any of the points $\{x_1, x_2, \ldots, x_n\}$. 
Nonstandard methods seem to be very helpful in proving this standard result. The picture below illustrates why. Suppose that the ball shown is standard, but that the part of $^*C$ shown is within $\delta$ of every point of $E$. 
Then this must repeat infinitely often with later portions of the curve. This allows us to create (many annoying details later) infinitely many disjoint regions that all are within $\delta$ of every point of $E$. 


