

# Towards Chainability For Certain Regions Obtained From Simple Dense Canals

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## Abstract

Nonstandard methods are used to obtain results about sets formed by simple dense canals in the plane. The first main result considers internal regions  $V$  that contain no standard points and have the property that  $st(V)$  does not disconnect the plane and are bounded by simple closed curves that have small intersection with  $^*st(V)$ . We show that such sets must be somewhat one-dimensional in the sense that they cannot contain a “Y-set” consisting of three arcs intersecting only at a single common point with the property that the non-intersecting endpoints of each arc are a noninfinitesimal distance from the arc joining the other two ends. The second main result is that such sets can always be formed by “closing off” sections of simple dense canals with arcs of infinitesimal length in such a way that the closed off portion passes within an infinitesimal distance of every point of the entire canal. The relevance of these results and techniques to the plane fixed point problem is discussed.

*Key words:* nonstandard topology, fixed points

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## 1 Preliminaries

We work in a nonstandard universe, as described for example in [3],[4],[6], or [7]. It is often most convenient to think of this setting as making use of a fairly strong set-theoretic language capable of expressing the mathematical statements of interest for some application and a standard universe that includes the usual sets, functions and relations of the current discourse. A parallel nonstandard universe is created in which the *transfer principle* applies, i.e. every statement expressible in our language that is true in the standard universe is true in the nonstandard universe and vice-versa. We work in a nonstandard model that is  $\aleph_1$ -saturated. Basically, this means that if we have a collection of mathematical statements of cardinality less than or equal

to that of the reals, that can be made within the set theory language we are working in, and this collection has the property that every finite collection of these statements is true, then the entire collection of them is true.

Our focus in this work is bounded sets in the standard and the nonstandard plane. Some of the basic nonstandard results that are used here include:

**Definition 1** *If  $p_1$  and  $p_2$  are points in the nonstandard plane we will write  $p_1 \approx p_2$  if they are an infinitesimal distance apart and we will say that  $p_1$  is **near**  $p_2$ .*

*If  $A$  is a standard set we will write  ${}^*A$  for its nonstandard counterpart.*

*If  $p$  is a point in the nonstandard plane that is within some standard distance of the origin we write  $st(p)$  for the unique standard point that is within an infinitesimal distance of  $p$ .*

*The elements of  ${}^*P(\mathbb{R}^2)$  are the **internal** sets (here we use  $P(A)$  to denote the power set of  $A$ ). Intuitively these are the subsets of  ${}^*\mathbb{R}^2$  that the nonstandard model recognizes as sets.*

**Proposition 1** *i) If  $A$  is any internal set in the plane then  $st(A)$  (the set of all standard points infinitesimally close to points in  $A$ ) is closed.*

*ii) If  $A$  is internal, connected and bounded (i.e. contained in some  ${}^*B(0, r)$  for a standard real  $r > 0$ ) in the nonstandard plane then  $st(A)$  is connected.*

**Proof.** i) If  $p$  is a limit point of  $st(A)$  then for any standard  $\varepsilon > 0$  there must exist points of  $A$  within  $\varepsilon$  of  $p$ . Internally we may let  $r = \inf_{a \in A} \|a - p\|$ , and  $r$  must be infinitesimal. But then  $p$  is a standard point near points in  $A$ , so  $p \in st(A)$ .

ii). Suppose that  $st(A)$  is not connected, and let  $U$  and  $V$  be a separation, i.e. suppose that  $st(A) \subset U \cup V$ , both are open, and both have nonempty intersection with  $st(A)$ . Let  $u \in U \cap st(A)$  and  $v \in V \cap st(A)$ . Then there exists  $a_u, a_v$  in  $A$  such that  $st(a_u) = u$  and  $st(a_v) = v$ . Since  $U$  and  $V$  are open, there exists a standard  $d > 0$  such that all points within  $d$  of  $u$  are in  $U$  and all points within  $d$  of  $v$  are in  $V$ . Thus, by transfer,  $a_u \in {}^*U$ , and  $a_v \in {}^*V$ , and these two sets are open and disjoint. Since  $A$  is connected and both  ${}^*U$  and  ${}^*V$  intersect  $A$ , there must exist a point  $x \in A$  that is not contained in  ${}^*U \cup {}^*V$ . Since  $A$  is bounded  $st(x)$  exists and since  $st(x)$  is an element of  $st(A)$ , it is contained in either  $U$  or  $V$ . As above, however, this means that  $x$  is in either  ${}^*U$  or  ${}^*V$ , and this contradiction completes the proof. ■

For convenience the definitions needed here are all collected below. The notion of “IL-chainable” is a natural extension of usual chainability definitions into this nonstandard setting. The two “Y-set” definitions do not seem to correspond exactly to other definitions in the literature. The notion of a simple dense canal is due to Sieklucki [9].

**Definition 2** *If  $a_1$  and  $a_2$  are two points in the plane or the nonstandard plane we will write  $L[a_1, a_2]$  for the straight line segment from  $a_1$  to  $a_2$ . If  $A$  is an arc (either standard or internal) and  $a_1$  and  $a_2$  are two points on  $A$  then we will write  $A[a_1, a_2]$  for the subarc from  $a_1$  to  $a_2$ .*

*We will write  $C(a, A)$  for the connected component of  $A$  containing  $a$ .*

*If  $S$  is a simple closed curve in the plane or the nonstandard plane we will write  $V_S$  for the bounded region whose boundary is  $S$  ( $V_S$  exists and is well-defined by the Jordan Curve Theorem).*

*By a **simple dense canal** in  $E$  we mean an infinite arc  $C$  contained in the complement of  $E$  such that  $C$  is the image of a one-to-one continuous function  $c$  from  $[0, \infty)$  to  $\mathbb{R}^2$  that has the property that for all  $\varepsilon > 0$  there exists  $t_0$  such that for all  $t \in [t_0, \infty)$ ,  $c(t)$  is on a “transverse cross cut” of distance less than  $\varepsilon$  (i.e. a segment whose endpoints are in  $E$  on opposite sides of the arc and whose length is less than  $\varepsilon$ ), and that is dense in  $E$  (i.e. is such that  $\overline{C} - C = E$ ).*

*We will call an internal subset of the nonstandard plane **IL-chainable** (for “infinitesimally linearly chainable”) if it can be covered by a hyperfinite collection of open sets  $G_0, \dots, G_n$  of infinitesimal diameter with the “linear chain” condition that  $G_i \cap G_j$  is nonempty iff  $|i - j| \leq 1$ .*

*If  $\delta > 0$  we will call a standard subset of the plane, or an internal subset of the nonstandard plane  **$\delta$ -chainable** if it can be covered by a finite (or hyperfinite) collection of open sets  $G_0, \dots, G_n$  of diameter less than  $\delta$  with the “linear chain” condition that  $G_i \cap G_j$  is nonempty iff  $|i - j| \leq 1$ .*

*We will say that a set  $A \subset {}^*\mathbb{R}^2$  contains a **Y-set** if there exist four points  $a, b, c$ , and  $x$  in  $A$ , arcs  $C_{ax}, C_{bx}$ , and  $C_{cx}$  in  $A$  intersecting only at  $x$ , from  $a$  to  $x$ ,  $b$  to  $x$ , and  $c$  to  $x$ , respectively, such that none of the points  $a, b$ , or  $c$  are infinitesimally close to any point on the arcs joining the others to  $x$  (thus, for example, no point of  $C_{ax}$  is infinitesimally close to  $b$  or to  $c$ ).*

*If  $\delta > 0$  we will say that a set  $A$  in the plane contains a **size  $\delta$  Y-set** if there exist four points  $a, b, c$ , and  $x$  in  $A$ , and arcs  $C_{ax}, C_{bx}$ , and  $C_{cx}$  in  $A$  intersecting only at  $x$  from  $a$  to  $x$ ,  $b$  to  $x$ , and  $c$  to  $x$ , respectively, such that none of the points  $a, b$ , or  $c$  are within  $\delta$  of any point on the arcs joining the*

others to  $x$  (thus, for example, no point of  $C_{ax}$  is within distance  $\delta$  of  $b$  or  $c$ ).

The term “simple dense canal” is sometimes used to denote an open set in the complement containing an arc  $C$  as above, rather than the arc itself.

We note that the IL-chainable sets that we are interested in will almost never be internally chainable in the classic sense that they could be covered by arbitrarily small chained sets in the nonstandard universe.

An important point about the definition of a simple dense canal is the requirement that the cross cuts originate on opposite sides of the arc. A more precise definition (as used for example in [8]) of a line segment being a “transverse cross cut” to  $C$  at the point  $c$  is that  $L \cap C = \{c\}$  and for a sufficiently small open disk neighborhood  $U$  of  $c$ , each component of  $U - L$  contains exactly one component of  $(U \cap C) - \{c\}$ . We are particularly interested here in regions obtained by connecting two points of the canal with small arcs in such a way that a simple closed curve is formed. In this setting opposite sides of the arc will literally mean inside versus outside the simple closed curve, at least for points not too near the added arc. In particular, if a nonstandard portion of the curve is connected in this way using an arc of infinitesimal length, then near any point on the curve that is more than an infinitesimal distance from that arc there are points in  ${}^*E$  contained inside this bounded region.

## 2 Fixed Points

The results in this paper are motivated by their relationship to the plane fixed point problem which can be stated as follows. If  $E$  is a non-separating plane continuum (i.e. a compact, connected subset of the plane whose complement is also connected) must it have the fixed point property? An extensive and comprehensive overview of results and questions related to the plane fixed point problem can be found in [2].

In 1967-1970 Bell, Sieklucki and Iliadis showed that if  $f$  is any fixed point free mapping on such a set  $E$  as above, then  $E$  contains a simple dense canal in a subset  $E'$  of the boundary of  $E$  such that  $f(E') = E'$  [1] [9] [5]. There is strong reason to believe that sets that arise naturally from such simple dense canals are IL-chainable. The value of this as a possible step toward the fixed point problem or open questions related to the problem is illustrated by the proposition below. This proof uses nonstandard methods but in a less fundamental way than the results of the next two sections, and can be rephrased in standard terms in a straightforward way. The ideas in the proof are similar to those in other other works related to fixed points (the concept

of trying to “trap” the image with chainable sets is very natural).

**Proposition 2** *With  $E$  and  $f$  as in the fixed point problem let  $D$  be a connected subset of  ${}^*E$  and  $R$  a connected open region containing  $D$  and suppose that the following conditions are satisfied:*

*i)  $R$  is IL-chainable, i.e.  $R \subset \cup_{k=0}^n G_k$  with each  $G_k$  of infinitesimal diameter and  $G_i \cap G_j$  nonempty iff  $|i - j| \leq 1$*

*ii) The boundary of  $R$ , except perhaps on  $R \cap G_0$  is contained in the complement of  $E$ , i.e.  $\partial R - G_0 \subset {}^*\mathbb{R}^2 - E$ .*

*iii) there exists an index  $j$  and a point  $p$  in  $D \cap G_j$  such that  $f(p) \in D - \cup_{k=0}^j G_k$ .*

*Then  $f$  has a fixed point.*

**Proof.** We first note that conditions i) and ii) imply that for each  $i$ , if  $p \in ({}^*E \cap \cup_{k=i+1}^n G_k) - G_i$  then  $\mathcal{C}(p, {}^*E - G_i) \subset {}^*E \cap \cup_{k=i+1}^n G_k$ . We may assume that  $D$  is a connected component of  ${}^*E$  in  $R$ , for if not we replace  $D$  by its connected component in  $R$  and the remainder of the conditions are still satisfied.

We first show that all points in  $D \cap G_j$  must be mapped into  $R - \cup_{k=0}^j G_k$ . If not, let  $q \in D \cap G_j$  such that  $f(q) \notin R - \cup_{k=0}^j G_k$ . Let  $\varepsilon > 0$  be an infinitesimal less than the distance between  ${}^*E \cap (R - \cup_{k=0}^i G_k)$  and any point in  ${}^*E - (R \cap (\cup_{k=i}^n G_k))$  (for  $i = 1, \dots, n$ ) (here we use condition ii) to see that such an  $\varepsilon$  exists). Now we let  $\delta > 0$  be the corresponding infinitesimal that exists by the uniform continuity of  $f$ . Since  $D$  is compact it can be covered by a (hyper)finite collection of balls of radius  $\delta/2$ , and since  $D$  is connected there is a sequence of these  $\delta/2$  balls such that the first ball in the sequence contains  $p$ , each ball in the sequence intersects the next, and the final ball contains  $q$ . We now note that if any point in any  $G_i \cap D$  is mapped to  $R - \cup_{k=0}^i G_k$ , then so is every point in  $G_i \cap D$  that is in the same  $\delta/2$  ball, since no point can move far enough to move out of the region, unless it were to move to  $G_i$  itself. But no point in  $G_i$  can map into  $G_i$  since the diameter is infinitesimal and there are no fixed points. But for the same reason, any point in an intersecting  $\delta/2$  ball must also stay inside  $R$  and move to a higher numbered set (again, using the fact that it cannot map into a region of the same index). Now by following the chain to the point  $q$  we get a contradiction.

Thus, every point in  $D \cap G_j$  must be mapped into  $R - \cup_{k=0}^j G_k$ . But  $D \cap G_j \cap G_{j+1}$  is not empty, or else we could disconnect  $D$  by using  $\cup_{k=0}^j G_k$  and  $\cup_{k=j+1}^n G_k$ , so there is a point in  $D \cap G_{j+1}$  that maps into  $R - \cup_{k=0}^j G_k$ , and since no point in  $G_{j+1}$  can map into a point in  $G_{j+1}$  there is a point that maps

into  $R - \cup_{k=0}^{j+1} G_k$ . Then, reasoning as above we see that all points in  $D \cap G_{j+1}$  map into  $R - \cup_{k=0}^{j+1} G_k$ , and continuing in this way we see that for all  $m > j$ , all points in  $D \cap G_m$  map into  $R - \cup_{k=0}^m G_k$ . Thus all points in  $D$  map into  $R$ , but since  $f(D)$  is connected,  $D$  is a connected component of  $*E$  in  $R$ , and  $f(p) \in D$ , we must have that  $f(D) \subset D$ . But from above this means that for each  $m$ , all points in  $D \cap G_m$  map into  $D - \cup_{k=0}^m G_k$ . This contradicts that  $n$  is (hyper)finite, completing the proof of the lemma. ■

With the result of Bell, Sieklucki and Iliadis in mind, we consider compact connected and co-connected sets  $E$  in the plane that contain a simple dense canal. We are especially interested in sets created by “closing off” a segment of  $C$  with a small arc to create a bounded region, and whether or not we can always obtain IL-chained sets as a result of this process. This question is still open, but in the next two sections we obtain a result that seems to be a significant step towards this goal.

### 3 Regions containing no standard points

In this section we make more fundamental use of the nonstandard setting, exploiting the interplay between a set and the internal set obtained by first taking the standard part and then looking at the nonstandard version of that set. Of particular importance are regions  $V$  that have little intersection with  $*st(V)$  and that contain no standard points. It is easy to find simple examples of regions  $V$  such that  $*st(V) \cap V$  is empty. For example if  $\eta > 0$  is infinitesimal and  $V$  is the interior of the rectangle with corners at  $(\eta, 1)$ ,  $(2\eta, 1)$ ,  $(2\eta, 0)$  and  $(\eta, 0)$  then  $st(V)$  is simply the line segment on the  $y$  axis from  $(0, 0)$  to  $(0, 1)$ . The set  $*st(V)$  is the nonstandard counterpart of  $st(V)$ , so includes nonstandard points on this line segment, including points within an infinitesimal of the origin, but since the standard universe satisfies the statement that all the points in  $st(V)$  have  $x$  coordinate of 0 the nonstandard version of this set has the same property by the transfer principle. This shows that  $*st(V) \cap V$  is empty. The boundary of  $V$  does not intersect  $*st(V)$  at all in this example. However, in the more complicated examples in which we are interested we will create regions by blocking off portions of simple dense canals, and these boundaries will always intersect  $*st(V)$ , although only on the small segments used to close off a region. There will always be many points of  $*st(V)$  in the interior of  $V$ , but the crucial additional condition in the theorem below is that none of them will be actual standard points.

**Theorem 3** *Let  $V \subset *R^2$  be a bounded (internal) region (i.e. contained in some  $*B(0, r)$  for a standard real  $r > 0$ ) bounded by a simple closed curve  $S$  and suppose that  $\bar{V}$  contains no standard points,  $st(V)$  does not disconnect the*

plane, and that there exists an arc  $A$  of infinitesimal length on  $S$  such that  $*st(V) \cap S \subset A$ . Then  $V$  contains no  $Y$ -set.

**Proof.** We first note that  $st(V)$  has no interior. Since  $\bar{V}$  contains no standard points, every point of  $*st(V)$  is near a point of  $\partial(V)$ . If  $*st(V)$  were to contain any ball of noninfinitesimal diameter, then it would intersect  $\partial V$  at points that are not all near each other, which violates the condition that  $*st(V) \cap \partial(V) \subset A$ . Thus,  $st(V)$  has no interior by transfer.

Suppose that  $V$  contains a  $Y$ -set. We let  $a, b, c$  and  $x$  be points as in the definition of a  $Y$ -set, but we note that there are many possible choices for  $a, b$ , and  $c$ , and since the arc  $A$  is of infinitesimal length, we may choose these points in such a way that none of them are near  $A$ . We let  $Y = C_{ax} \cup C_{bx} \cup C_{cx}$ .

We next note that there must exist standard disjoint balls  $B_a, B_b$ , and  $B_c$  satisfying the following:

- i)  $a \in *B_a, b \in *B_b$ , and  $c \in *B_c$ .
- ii) Each of these three balls have non-infinitesimal radius.
- iii) The arc  $A$  does not intersect  $\bar{B}_a, \bar{B}_b$ , or  $\bar{B}_c$
- iv) every point on  $\bar{B}_a$  is not near any point on  $C_{bx}$  or  $C_{cx}$ , every point on  $\bar{B}_b$  is not near any point on  $C_{ax}$  or  $C_{cx}$ , and every point on  $\bar{B}_c$  is not near any point on  $C_{ax}$  or  $C_{bx}$
- v) The interior of each ball contains points not disconnected from infinity by  $st(Y)$  together with the closure of the other two balls. For example, the interior of  $B_a$  contains points not disconnected from infinity by  $st(Y) \cup \bar{B}_b \cup \bar{B}_c$ .

It is clear that standard balls of sufficiently small radius can be found that will satisfy the first four properties, and we assume that we begin with a  $B_a^0, B_b^0$ , and  $B_c^0$  satisfying just those. It is possible that, for example,  $st(Y) \cup \bar{B}_b^0 \cup \bar{B}_c^0$  disconnects the plane. However, since  $st(V)$  does not disconnect the plane, and no point of  $\bar{B}_b^0$  is near  $\bar{B}_c^0$  or any point on  $C_{ax}$  or  $C_{cx}$ , a bounded component of the complement of  $st(Y) \cup \bar{B}_b^0 \cup \bar{B}_c^0$  could occur only if a portion of  $C_{bx}$  encloses some standard open set and then passes back into or within an infinitesimal of  $\bar{B}_b^0$  or if a portion of  $C_{cx}$  encloses some standard open set and then passes back into or within an infinitesimal of  $\bar{B}_c^0$ . If  $B_a^0$  is in such a bounded component (all points in this set must be in the same component since no point of the boundary intersects  $B_a^0$ ) we let  $p_a$  be a standard point in  $B_a^0 - st(V)$ , and  $T_a$  be a standard path from  $p_a$  to infinity in  $\mathbb{R}^2 - (st(V))$ . This standard path cannot pass near  $b$  or  $c$  since it stays a standard nonzero distance away from  $st(V)$ . Similarly if  $B_b^0$  is in a bounded component of  $st(Y) \cup \bar{B}_a^0 \cup \bar{B}_c^0$  we let  $p_b$  be a standard point in  $B_b^0 - st(V)$ , and  $T_b$  be a standard path from  $p_a$  to

infinity in  $\mathbb{R}^2 - (st(V))$ , and if  $B_c^0$  is in a bounded component of  $st(Y) \cup \overline{B_a^0} \cup \overline{B_b^0}$  we let  $p_c$  be a standard point in  $B_c^0 - st(V)$ , and  $T_c$  be a standard path from  $p_c$  to infinity in  $\mathbb{R}^2 - (st(V))$ . We can now choose smaller balls  $B_a, B_b$ , and  $B_c$  inside  $B_a^0, B_b^0$ , and  $B_c^0$  such that conditions i) through iv) are satisfied and none of these intersect  $T_a, T_b$  or  $T_c$ . This ensures that condition v) will hold, for points in  $(\mathbb{R}^2 - (st(V))) \cap B_a$  are in the same connected component of  $st(Y) \cup \overline{B_b} \cup \overline{B_c}$  as  $p_a$ , points in  $(\mathbb{R}^2 - (st(V))) \cap B_b$  are in the same connected component of  $st(Y) \cup \overline{B_a} \cup \overline{B_c}$  as  $p_b$  and points in  $(\mathbb{R}^2 - (st(V))) \cap B_c$  are in the same connected component of  $st(Y) \cup \overline{B_a} \cup \overline{B_b}$  as  $p_c$ .

We now let  $a'$  be the last point of  $C_{ax} \cap \overline{B}$  on the arc from  $a$  to  $x$ , and define  $C_{a'x}$  to be the subarc of  $C_{ax}$  from  $a'$  to  $x$ . We define  $b', C_{b'x}, c', C_{c'x}$  analogously. We let  $Y' = C_{a'x} \cup C_{b'x} \cup C_{c'x}$ , and we note that  $Y'$  is path-connected and therefore connected. By proposition 1  $st(Y')$  is connected, and thus  $*st(Y')$  is connected by transfer.

We let  $q_a$  be a standard point in  $B_a - st(V)$  and  $T'_a$  be a standard infinite path (ray) in  $\mathbb{R}^2 - (st(Y) \cup \overline{B_b} \cup \overline{B_c})$  starting at  $q_a$  and going off to infinity. Similarly we let  $q_b$  be a standard point in  $B_b - st(V)$  and  $T'_b$  be a standard infinite path in  $\mathbb{R}^2 - (st(Y) \cup \overline{B_b} \cup \overline{B_c} \cup T'_a)$  starting at  $q_b$  and going off to infinity, and  $q_c$  be a standard point in  $B_c - st(V)$  and  $T'_c$  be a standard infinite path (ray) in  $\mathbb{R}^2 - (st(Y) \cup \overline{B_b} \cup \overline{B_c} \cup T'_a \cup T'_b)$  starting at  $q_c$  and going off to infinity.

Since  $st(Y')$  and any bounded portion of  $T'_a, T'_b$ , and  $T'_c$  are disjoint compact sets, they are some non-zero distance apart, so by transfer there are no points of  $*st(Y')$  that are within an infinitesimal distance of any points of  $*T'_a, *T'_b$ , or  $*T'_c$ . We note that if  $*st(Y') \cap S$  is nonempty, then the single point  $st(A)$  is in  $st(Y')$ , so the arc  $A$  is also not within an infinitesimal distance of any points of  $*T'_a, *T'_b$ , or  $*T'_c$ . In particular, none of these arcs will intersect the arc  $A$  unless the arc  $A$  has empty intersection with  $*st(Y')$ .

We now define internal infinite arcs as follows: We let  $\Gamma_a$  consist of the line segment from  $a'$  to the nearest point of  $*T'_a \cap \overline{B_a}$ , followed by  $*T'_a$  from that point to infinity. The internal paths  $\Gamma_b$  and  $\Gamma_c$  are defined analogously. Since there are no points of  $*st(Y')$  in the interior of  $B_a, B_b$ , and  $B_c$  there are no points of  $*st(Y')$  other than  $a'$  that intersect  $\Gamma_a$ , and similarly the other two arcs only intersect  $*st(Y')$  at their initial points. Since the arc  $A$  does not intersect  $\overline{B_a}, \overline{B_b}$ , or  $\overline{B_c}$  we also know that  $A$  can only intersect  $\Gamma_a, \Gamma_b$ , and  $\Gamma_c$  on  $*T'_a, *T'_b$ , and  $*T'_c$ , and this can only occur when  $*st(Y') \cap A$  is empty.

The set  $\Gamma_a \cup \Gamma_b \cup \Gamma_c \cup Y'$  consists of three non-intersecting infinite arcs originating from the point  $x$ , and so divides the nonstandard plane into three regions. One of these regions, which we will call  $R_1$ , is bounded by  $\Gamma_a \cup C_{a'x} \cup C_{b'x} \cup \Gamma_b$ . No point in this region can be within an infinitesimal distance

of  $c'$ , since  $c'$  is not contained in  $R_1$  and is not near any of the points on the boundary. Similarly we let  $R_2$  be the region bounded by  $\Gamma_b \cup C_{b'x} \cup C_{c'x} \cup \Gamma_c$  and  $R_3$  be the region bounded by  $\Gamma_c \cup C_{c'x} \cup C_{a'x} \cup \Gamma_a$ , noting that no point of  $R_2$  is near  $a'$  and no point of  $R_3$  is near  $b'$ .

We consider connected components of  ${}^*\mathbb{R}^2 - (S \cup \Gamma_a \cup \Gamma_b \cup \Gamma_c)$ . One of these is  $V$ , which intersects all three of our newly constructed regions, but it is not difficult to see that this is the only connected component of this set that can intersect more than one of these regions. This is because other connected components of this set are bounded by the outside of  $S$  together with portions of the  $\Gamma$  curves, and since  $S$  does not intersect  $Y$ , a point traveling along  $S$  may only move from one of the three regions into another one by passing through  $\Gamma_a$ ,  $\Gamma_b$ , or  $\Gamma_c$ .

No standard points of  ${}^*st(Y')$  can be on any of the boundary points of  $R_1$ ,  $R_2$ , or  $R_3$ , so all of the standard points of this set are in one of the three regions  $R_1$ ,  $R_2$ , or  $R_3$ . We will assume, wlog, that there is a standard point  $p_1$  in  $R_1$ . Since  $p_1$  is a standard point it cannot be in  $V$ , so from above  $\mathcal{C}(p_1, {}^*\mathbb{R}^2 - (S \cup \Gamma_a \cup \Gamma_b))$  must be contained in  $R_1$ . But  $st(c')$  cannot be in  $R_1$  since no point in  $R_1$  is within an infinitesimal distance of  $c'$ , and so wlog we suppose that it is in  $R_2$ . By a similar argument,  $\mathcal{C}(st(c'), {}^*\mathbb{R}^2 - (S \cup \Gamma_b \cup \Gamma_c))$  is contained in  $R_2$ . But at least one of these two sets has its entire boundary contained in  ${}^*\mathbb{R}^2 - {}^*st(Y')$ . To see this we note that the only possible point of intersection of  ${}^*st(Y')$  with either boundary must occur on the arc  $A$  of infinitesimal length, and none of the three  $\Gamma$  curves intersect with  $A$  unless  ${}^*st(Y') \cap A$  is empty. Thus, the arc  $A$  cannot cross from one region to another unless it is essentially irrelevant (i.e. cannot contain points of  ${}^*st(Y')$ ). It could then be a part of the boundary of one of these two sets, but not both. Thus, these two regions are in different connected components of  ${}^*\mathbb{R}^2 - {}^*st(Y')$  as well, contradicting the fact that  ${}^*st(Y')$  is connected, and this contradiction completes the proof. ■

It is easy to see that the condition that  $\bar{V}$  contains no standard points is necessary, for without it we could let  $Y$  be the set in the nonstandard plane consisting of the closed line segments from  $(-1, 0)$  to  $(1, 0)$  and  $(0, 0)$  to  $(0, 1)$ , and let  $V$  be the interior of a simple closed curve that traces around  $Y$ , and stays infinitesimally close to it at all points. Such a  $V$  satisfies the other conditions of the proposition (the boundary of  $V$  does not intersect  ${}^*st(V)$  at all), but certainly contains the  $Y$ -set that we surrounded. Similarly, if  $\eta$  is infinitesimal we can surround the set consisting of the line segments from  $(-1, \eta)$  to  $(1, \eta)$  together with the segment from  $(\eta, \eta)$  to  $(\eta, 1)$  with a simple closed curve containing no standard points if we allow the curve to have two

disjoint arcs of infinitesimal length intersect the standard part, rather than only one, as in the theorem. In this case the two disjoint arcs can even be infinitesimally close to each other (though not close as one follows along the curve).

As discussed in more detail in the last section, it is hoped that the theorem can be improved to allow us to conclude that the set  $V$  is IL-chainable. It is not known whether or not the condition that  $V$  contains no Y-set implies that  $V$  is IL-chainable. The proposition below seems to suggest that this might not be the case.

**Proposition 4** *If  $n$  is any natural number and  $\delta > 0$  then there exist sets  $X$  which contain no size  $\delta$  Y-set but are not  $n\delta$  chainable.*

**Proof.** Given  $\delta$  and  $n$  we let  $d$  be greater than  $n\delta$  and let the set  $X$  consist of the graph of  $y = (d/\delta)x |\sin((\pi/\delta)x)|$  on  $[0, d]$  together with the set of all line segments of the form  $L[(\delta/2+k\delta, d/2+kd), (\delta/2+k\delta, d/2+(k+1)d)]$  for which  $\delta/2+k\delta < d$ . These vertical line segments of length  $d$  connect to the sine part of the graph at points where the curve intersects the line  $y = (d/\delta)x$ . The set  $X$  does not contain a size  $\delta$  Y-set. To see this we note that the only possible points  $x$  in the definition of a size  $\delta$  Y-set are of the form  $(\delta/2+k\delta, d/2+kd)$  and if the arc that follows the sine curve in the positive direction from that point extends at least to the next time the curve intersects the line  $y = (d/\delta)x$  then every point on the vertical segment above such an  $x$  is within  $\delta$  of that portion. Any portion beyond such an  $x$  that includes less than one full period is itself within  $\delta$  of either the line segment or previous portions of the curve. But this set is not  $n\delta$  chainable, for the line segments cannot be contained in a single chaining set, and the points of the form  $(\delta/2+k\delta, d/2+kd)$  would force an intersection of three covering sets unless the points on the next loop of the curve are contained in the same set. This will force a width of chaining sets equal to the entire range of the set in the  $x$  direction. ■

To obtain an example that involves a region we may surround the set  $X$  in the above example with a simple closed curve very near to  $X$  at all points and consider the bounded interior set. Perhaps this example can be improved to show that there exists a region with no Y-set that fails to be IL-chainable, but it does not appear to be straightforward. One problem is that extending this in the simplest possible way as  $n$  increases yields a graph that has height greater than any standard natural number within a bounded  $x$  domain, and we are interested here in examples in which the standard part of all points exists, i.e. in sets contained inside some standard disk.

It seems quite likely that the theorem can be improved to show that the set  $V$  is IL-chainable even if this does not follow from the condition that it contain no

Y-set. This is because the condition that  $V$  contains no standard points and has small intersection with  $*st(V)$  is quite strong and precludes, for example, the pursuit of any counterexample in which there are sets similar to portions of the graph of  $\sin(Kx)$  where  $K$  is in  $*\mathbb{N} - \mathbb{N}$  over some non-infinitesimal domain. The standard part of such a set would contain a standard rectangle along with its interior. Such sets with long patches of many thin long loops seem to be the most promising examples of sets with no Y-sets that might not be IL-chainable, but the intersection of  $\partial V$  with  $*st(V)$  is large in such cases.

#### 4 Regions Defined by portions of a simple dense canal

Our goal in this section is to show that sets that arise in conjunction with simple dense canals satisfy the conditions of theorem 3.

**Theorem 5** *Let  $E$  be a non-separating plane continuum that contains a simple dense canal  $C$ . Then there exist points  $p_1 \approx p_2$  on  $*C$  and an arc  $A$  of a circle such that the portion of  $*C[p_1, p_2]$  together with  $A[p_2, p_1]$  forms a simple closed curve  $S$  that is infinitesimally close to every point in  $E$  and is such that  $V_S$  contains no standard points in  $E$ .*

We note that every point on  $*C - C$  has standard part in  $E$  since it lies on a crosscut of infinitesimal length. We will let  $c_0$  be the initial point of  $C$  and if two points  $u$  and  $v$  are on  $C$  we will write  $u \prec v$  to indicate that  $u$  is earlier in the curve than  $v$ , i.e. that  $u \in C[c_0, v]$ .

Before proving the theorem we first need the following lemma.

**Lemma 6** *Let  $E$  and  $C$  be as in the theorem, let  $\delta > 0$  be a standard real number, and suppose that there exist points  $u$  and  $v$  in  $*C - C$  such that the following conditions are satisfied;*

- i) both  $u$  and  $v$  are on the boundary of a standard disk  $D$  of radius less than  $\delta$ .*
- ii)  $*C[u, v]$  is within  $\delta$  of every point in  $E$ .*
- iii)  $*C[u, v] \cap \overline{D} = \{u, v\}$ .*

*We note that condition iii) implies that  $*C[u, v]$  together with  $L[v, u]$  forms a simple closed curve, which we will call  $S$ .*

*Then one of the following must be true:*

a) arbitrarily far out on  $*C$  and arbitrarily close to  $st(u)$  and  $st(v)$ , respectively, there exist points  $u'$  and  $v'$  such that conditions i) through iii) are satisfied if we replace  $u$  and  $v$  with  $u'$  and  $v'$  in each statement, and the simple closed curve  $S'$  (formed by  $*C[u', v']$  together with  $L[v', u']$ ) is contained in  $V_S$  (the “inside” region defined by  $S$ ) or

b) arbitrarily far out on  $*C$  and arbitrarily close to  $st(u)$  and  $st(v)$  there exist points  $u'$  and  $v'$  satisfying all the same conditions except that  $S$  is contained in  $V_{S'}$ .

**Proof.** The standard set  $F = st(*C[u, v])$  is a closed, connected subset of  $E$ . We note that no element of  $F$  (and hence no element of  $*F$  by transfer) intersects the interior of  $D$  and all points on  $S$  other than those in the interior of  $D$  are in the complement of  $E$ . Since  $*F$  is connected and does not intersect  $S$  we see that either  $*F \subset V_S$  or  $*F \cap V_S = \emptyset$ . These two cases will determine which of a) or b) is true in the lemma.

Case 1)  $*F \subset V_S$ .

Let  $x$  be some standard point in  $*F$ . So,  $x \in V_S$  and is infinitesimally close to but not contained in  $*C[u, v]$  (since  $x$  is in  $E$ ). We let  $\eta$  be a positive infinitesimal less than the minimum distance between  $x$  and  $*C[u, v]$ . Then in the nonstandard universe  $*C[u, v]$  is witness to the fact that for any standard  $\epsilon > 0$  and beyond any standard point in  $C$  there exist points  $u$  and  $v$  on the curve such that properties i) through iii) hold, and such that  $x$  is in  $V_S$ ,  $x$  is within  $\epsilon$  of  $*C[u, v]$  and  $u$  and  $v$  are within  $\epsilon$  of  $st(u)$  and  $st(v)$  respectively. Thus, this is true in the standard universe as well by transfer, and so the standard universe satisfies the statement that there are points arbitrarily far out on  $C$  such that i)-iii) hold and such that the relevant portion of the curve passes arbitrarily close to  $x$ . Then, again by transfer, this is true in the nonstandard universe, and so beyond any point on the curve  $*C$  there exist points  $u'$  and  $v'$  satisfying conditions i) through iii) and such that  $x$  is in  $V_S$ , and such that the minimum distance from  $*C[u', v']$  to  $x$  is less than  $\eta$ , and  $u'$  and  $v'$  are within an infinitesimal distance of  $st(u)$  and  $st(v)$  respectively. These conditions now guarantee that  $S'$  is contained in  $V_S$ , and condition a) is satisfied.

Case 2)  $*F \cap V_S = \emptyset$ . We again let  $x$  be some standard point in  $*F$ , noting that this time  $x$  is outside of  $V_S$  and infinitesimally close to  $*C[u, v]$ . We again define  $\eta$  as in case 1) and follow a completely analogous argument to obtain the existence of the points  $u'$  and  $v'$  satisfying all the same conditions as in case 1) except that  $x$  is not in  $V_{S'}$ . These conditions now guarantee that  $S$  is contained in  $V_{S'}$ . ■

**Proof.** (of the theorem). By saturation it suffices to show that for any finite collection of standard points  $\{x_1, x_2, \dots, x_n\}$  in  $E$  and any standard  $\delta > 0$  there

exist points  $a$  and  $b$  on  $C$  and an arc of some circle  $A$  such that  $C[a, b]$  is within  $\delta$  of every point of  $E$ ,  $a$  and  $b$  are within  $\delta$  of each other, and the portion of  $C$  from  $a$  to  $b$  together with  $A[b, a]$  forms a simple closed curve whose interior does not contain any of the points  $\{x_1, x_2, \dots, x_n\}$ .

Let  $z$  be a point on  $*C - C$ . We cover  $E$  with finitely many standard disks  $\{D_1, D_2, \dots, D_m\}$  of radius less than  $\delta/2$ . As we travel backwards on  $*C$  from the point  $z$  there will be a last disk with the property that its closure is intersected by the arc. We re-number to let that disk be  $D_1$  and let the first point of intersection of  $*C$  and  $*D_1$  (as we travel backward from  $z$ ) be denoted by  $a_1$ . Thus  $*C[a_1, z]$  intersects the boundary of  $D_1$  but not the interior, and intersects the closure of all of the other  $D_i$ 's. By the density condition on  $C$  and the fact that there are points in  $E$  in the interior of  $D_1$  there exist points further on the arc than  $z$  that intersect  $\overline{D_1}$ , and we let  $b_1$  be the first such. We will end up using a portion of  $\partial D_1$  to be our arc  $A$ , so for simplicity we will write  $A$  for  $\partial D_1$ . Now  $L[b_1, a_1]$  only intersects  $*C[a_1, b_1]$  at  $a_1$  and  $b_1$  and so forms a simple closed curve  $S_1$  when joined with  $*C[a_1, b_1]$ .

The conditions of the lemma are now satisfied, with  $a_1, b_1$  and  $D_1$  replacing  $u, v$  and  $D$  in the statement. We note that the portion of  $*C$  beyond every finite point cannot intersect a standard point, so that  $a_1 \neq st(a_1)$  and  $b_1 \neq st(b_1)$ . It is conceivable that  $st(a_1) = st(b_1)$ .  $*C$  will intersect  $A[a_1, st(a_1)]$  arbitrarily far out and arbitrarily close to  $st(a_1)$  by the lemma, and we will denote the next point of intersection of  $*C$  and  $A[a_1, st(a_1)]$  after  $b_1$  by  $c_1$ . We let  $a_0$  be the first point of  $*C$  to intersect  $A[a_1, c_1]$ , noting that  $a_0 \preceq a_1$ . Using the lemma once again, we see that there exists a point  $a_2$  on  $A \cap *C$  nearer to  $st(a_1)$  than any point on  $*C[a_0, c_1] \cap A$  and a point  $b_2$  such that conditions i) through iii) of the lemma are satisfied when  $u, v$  and  $D$  are replaced by  $a_2, b_2$  and  $D_1$ .

We let  $d_1$  be the last point of intersection of  $*C[c_1, a_2]$  on  $A[a_0, c_1]$  (thus  $c_1 \preceq d_1$ ). We note that the simple closed curve formed by  $*C[a_0, d_1]$  together with  $A[d_1, a_0]$  divides the plane into two sets, both of which contain elements of  $*E$ , and so by the density condition on the curve,  $*C$  must intersect  $A[d_1, a_0]$  arbitrarily far out. We define  $c_2$  to be the next point on  $*C$  after  $b_2$  that intersects  $A[d_1, a_0]$ . Then  $*C[d_1, c_2]$  together with  $A[c_2, d_1]$  forms a simple closed curve  $S_1$  and we will write simply  $V_1$  for the corresponding region  $V_{S_1}$ .

We want to continue with these inductive definitions, but with an essential change. If  $V_1$  does not contain  $st(a_1)$  then  $A[a_0, c_2]$  together with  $*C[a_0, c_2]$  disconnects  $st(a_1)$  from  $V_1$ . This is the case regardless of whether  $st(a_1)$  is in the bounded region bounded by this curve or not. In this case we will want to define  $d_2$  analogously to  $d_1$ , i.e. let it be the nearest point to  $a_0$  on  $A[a_0, c_2] \cap *C[c_2, a_3]$ . Since  $a_3$  will be chosen to be nearer to  $st(a_1)$  than any point on  $*C[a_0, c_2]$  there will be no point between  $d_2$  and  $a_3$  on  $*C$  that is

not in whichever region bounded by  $*C[a_0, d_2]$  together with  $A[d_2, a_0]$  contains  $st(a_1)$ , for if there were the curve would need to cross  $A[a_0, d_2]$  again, creating a later point than  $d_2$  on  $A[a_0, c_2] \cap *C[c_1, a_3]$ . However, if  $V_1$  does contain  $st(a_1)$  we must replace the segment  $A[a_0, c_2]$  in this definition with the segment  $A[d_1, c_2]$  in order to maintain the analogous desirable property that provides some control on the segment of the curve between  $d_2$  and  $a_3$ . In order to do this we need to add one more set of terms  $x_k, y_k$  to our inductive definition that essentially just keeps track of where the “doorway” into the inner region containing  $st(a_1)$  is. At each new level the curve must get inside that region in order to be closer to  $st(a_1)$  than any of the previous parts of the curve. By beginning and ending our curves of interest at this segment we can achieve the desired disjointness property.

So, assuming that we have defined  $a_0, a_1, a_2, b_1, b_2, c_1, c_2, d_1$  and  $V_1$  as above, we let  $x_1 = a_0, y_1 = c_1$  and make the following induction definitions:

$$x_{k+1} = x_k \text{ and } y_{k+1} = c_{k+1} \text{ if } st(a_1) \notin V_k$$

$$x_{k+1} = d_k \text{ and } y_{k+1} = c_{k+1} \text{ if } st(a_1) \in V_k$$

$a_{k+1}, b_{k+1}$  are the next points on  $*C \cap A$  after  $c_k$  such that conditions i) through iii) of lemma 2 are satisfied when  $u, v$  and  $D$  are replaced by  $a_{k+1}, b_{k+1}$  and  $D_1$ , respectively, and such that  $a_{k+1}$  is nearer to  $st(a_1)$  than any point on  $*C[a_0, c_k] \cap A$ .

$$d_k = \text{the last point of intersection of } *C[c_k, a_{k+1}] \text{ on } A[x_k, y_k]$$

$$c_{k+1} = \text{the first point after } b_{k+1} \text{ on } *C \text{ to intersect the segment } A[x_k, d_k]$$

$S_k = \text{the simple closed curve formed by } *C[d_k, c_{k+1}] \text{ together with } A[c_{k+1}, d_k]$   
and

$$V_i = V_{S_i}.$$

We note that no point  $x$  on  $*C$  with  $x \prec a_0$  intersects  $A[x_i, y_i]$  for any  $i$ , and no point  $x$  on  $*C$  with  $x \prec c_{i+1}$  intersects any  $A[d_i, c_{i+1}]$ , except for  $d_i$  itself. We also note that the curve  $C$  starts some standard distance away from  $E$  and every point in any  $V_i$  is within an infinitesimal distance of points in  $E$ . These observations show that  $S_i$  is well-defined, and that no  $V_i$  can contain  $a_0$ , or any boundary point of any earlier  $V_j$  since tracing backward along the curve  $C$  we would obtain an intersection point with  $L[d_i, c_{i+1}]$  that violates the condition above. No  $V_i$  that does not contain  $st(a_1)$  can contain any boundary point of any later  $V_j$ , for if the curve  $*C$  enters into  $V_i$  it would have to pass back through  $A[x_j, y_j]$  later than  $d_j$  in order to re-enter  $V_j$ . Since each  $V_i$  contains a boundary portion consisting of  $*C[a_{i+1}, b_{i+1}]$  they all have the property that they pass within  $\delta$  of all points in  $E$ . Thus, if there are infinitely many  $V_i$  that

do not contain  $st(a_1)$  then we have achieved our desired result of an infinite disjoint collection of such regions.

So, we assume that beyond some index  $m$  all  $V_i$  contain  $st(a_1)$ , so that for all  $i > m$   $x_{i+1} = d_i$ ,  $y_{i+1} = c_{i+1}$  and  $V_{i+1} \subset V_i$ . Then each later time that the curve passes through the segment  $A[x_k, y_k]$  at  $d_k$  it must pass through  $A[b_1, st(b_1)]$  before it can reach the points near  $st(a_1)$ , and thus before it can reach  $a_{k+1}$ . There are points arbitrarily close to  $st(a_1)$  that are connected by much later portions of  $*C$  to points very close to  $st(b_1)$ , and no portion of the curve that forms the boundary of earlier  $V_i$ 's can intersect this. Thus, the fact that  $st(a_1)$  is contained in each  $V_i$  for  $i > m$  means that  $st(b_1)$  and these much later portions of the curve are as well. For each  $i > m$  the portion of  $*C$  after  $d_i$  must cross  $A[b_1, st(b_1)]$ , then cross  $A$  at a point nearer to  $st(a_1)$  than any point on  $*C$  before  $c_i$  (at the point  $a_{i+1}$ ) then intersect  $A$  at  $b_{i+1}$  and exit back through  $A[b_1, st(b_1)]$  and then finally intersect  $A[x_i, d_i]$ . We let  $u_k$  be the last point on  $*C \cap A[b_1, st(b_1)]$  before  $a_{k+1}$ . Then there are no points of  $*C[u_k, b_{k+1}]$  on  $A[b_1, st(b_1)]$  except  $b_{k+1}$  and  $u_{k+1}$ , so  $*C[u_k, b_{k+1}] \cup A[b_{k+1}, u_k]$  forms a simple closed curve  $T_k$ .

For  $k > m$ , every time  $*C$  enters a new  $V_k$  at  $d_k$  the side of  $*C$  on which it will need to ultimately exit (i.e. reach  $c_{k+1}$ ) switches (this side is also the eventual inside direction of  $V_k$ ). Thus, for half of the indices  $k > m$ , the first point at which  $*C$  intersects  $A[b_1, st(b_1)]$  after  $d_k$  will have its inside edge on the  $b_1$  side. We focus only on those  $k$ , and consider two cases:

i)  $V_{T_k}$  contains  $st(a_1)$ . Then the curve will not be able to exit to  $c_{k+1}$  in such a way that  $st(a_1)$  is contained inside unless it intersects  $A[a_{k+1}, st(a_1)]$  before it reaches  $c_{k+1}$  and it must do so with  $st(a_1)$  on the inside direction. We let  $v_k$  be the nearest such point to  $a_{k+1}$  that occurs on  $*C[b_{k+1}, c_{k+1}]$  and in this case we let  $U_k$  be the bounded region bounded by  $*C[a_{k+1}, v_k] \cup A[v_k, a_{k+1}]$ .

ii)  $V_{T_k}$  does not contain  $st(a_1)$ . In this case we let  $v_k$  be the last point of  $*C[b_{k+1}, c_{k+1}] \cap A[b_{k+1}, u_k]$ , and let  $U_k$  be the bounded region bounded by  $*C[u_k, v_k] \cup A[v_k, u_k]$ .

In this way we can obtain an infinite set of regions  $U_k$  that all contain  $*C[a_{k+1}, b_{k+1}]$  as parts of their boundaries, so these sets also have the property that they pass within  $\delta$  of all points in  $E$ . Each of the regions  $U_k$  are carved out of portions of  $V_{k-1}$ , as all of the boundary points of  $U_k$  are contained inside  $V_{k-1}$ . None of the  $U_k$  contain  $st(a_1)$  by the construction, and all the inside boundaries of the  $U_k$  consist of outside boundaries of  $V_k$  so every  $U_k \cap V_k$  is empty. If  $j > k$  then  $U_j \subset V_k$  so that  $U_k \cap U_j$  is empty. Thus, the case in which we are not able to construct an infinite disjoint collection of regions whose boundary passes within  $\delta$  of every point of  $E$  using the  $V_i$  regions allows the construction of an infinite collection of the  $U_i$  sets that have this property. The portions of  $A$

that are used in all of these sets are infinitesimal, and so less than  $\delta$  in length.

This completes the proof, for now given any finite collection of standard points  $\{x_1, x_2, \dots, x_n\}$  in  $E$ , one of the  $V_i$  or  $U_i$  from above must not contain any of these points, and we may use the portion of the curve that created that  $V_i$  or  $U_i$  to satisfy the conclusion of the theorem. ■

These two results now give us the standard theorem below.

**Theorem 7** *Let  $E$  be a non-separating plane continuum that contains a simple dense canal, and let  $C$  be an infinite arc (i.e. the image of a one-to-one continuous function from  $[0, \infty)$  to  $\mathbb{R}^2$ ) in the complement of  $E$  with the property that  $\overline{C} - C = E$  and for every  $\varepsilon > 0$  there exists a point  $p$  on  $C$  such*

*that all points on the arc beyond  $p$  are on a transverse cross cut of distance less than  $\varepsilon$ . Then for any  $\delta > 0$  there exist points  $p_1$  and  $p_2$  on  $C$ , and an arc of a circle  $A$ , such that  $C[p_1, p_2]$  together with  $A[p_1, p_2]$  forms a simple closed curve  $S$  that is within  $\delta$  of every point in  $E$  and is such that  $V_S$  contains no size  $\delta$  Y-set.*

**Proof.** Suppose that the theorem is not true for some  $E$ ,  $C$  and  $\delta > 0$ . By theorem 5 there exist points  $p_1 \approx p_2$  on  ${}^*C$  and an arc  $A$  of a circle such that the portion of  ${}^*C[p_1, p_2]$  together with  $A[p_2, p_1]$  forms a simple closed curve  $S$  that is infinitesimally close to every point in  $E$  and is such that  $V_S$  contains no standard points in  $E$ . The conditions of theorem 3 are satisfied for  $V_S$  since its boundary consists of points in the complement of  ${}^*E$  except on an arc of infinitesimal length, and since  $st(V_S) = E$  which does not disconnect the plane. Thus,  $V_S$  contains no Y-set. However, the statement that every such region formed in this way that is within  $\delta$  of every point in  $E$  contains a size  $\delta$  Y set is expressible, and we are assuming it is true in the standard universe. Thus, it is true in the nonstandard world as well by transfer, and this contradiction completes the proof. ■

#### 4.1 Questions and Conjectures

The conjecture is that theorem 3 can be improved to the statement below.

**Conjecture 8** *Let  $V \subset {}^*\mathbb{R}^2$  be a bounded region (i.e. contained in some  ${}^*B(0, r)$  for a standard real  $r > 0$ ) bounded by a simple closed curve (internally), and suppose that  $\overline{V}$  contains no standard points,  $st(V)$  does not disconnect the plane, and that there exists an arc  $A$  of infinitesimal length on  $\partial(V)$  such that  ${}^*st(V) \cap \partial(V) \subset A$ . Then  $V$  is IL-chainable.*

If the conjecture is true then the conclusion of theorem 7 can be improved to the condition that  $V_S$  is  $\delta$ -chainable. Assuming that it is true, a particularly important next question would be how much control it might be possible to achieve about where the arc is contained in terms of the chaining sets. Is it always possible for example to find regions defined by portions of a simple dense canal that are infinitesimally close to every point in  $E$  and which have the property that the arc used to close off the region is at one end of the chain? Of particular importance would be conditions under which it is possible to show that whenever  $f$  is a continuous function from  $E$  to  $E$  there is such a  $V_S$  that is IL-chainable and a point  $p$  in  $E$  such that both  $p$  and  $f(p)$  are in the same connected component of  $*E \cap V_S$  and the chaining set containing  $p$  is between the chaining set containing  $f(p)$  and the chaining set or sets that contain the infinitesimal arc used to close off  $V_S$ . By proposition 1 this would provide a positive solution to the plane fixed point problem under these conditions.

## References

- [1] Bell, H., "On fixed point properties of plane continua", *Trans. of the Amer. Math. Soc.*, 1967, **128**, 539-548
- [2] Hagopian, C.L., "An update on the elusive fixed-point property", in *Open Problems in Topology II*, ed. by E. Pearl, Elsevier, 2007
- [3] Henson, C.W., "Foundations of nonstandard analysis-A gentle introduction to nonstandard extensions", in *Nonstandard Analysis: Theory and Applications*, ed. by N.J. Cutland, C.W. Henson, and L. Arkeryd, Kluwer Academic Publishers, 1997
- [4] Hurd, A.E., and Loeb, P.A., *An Introduction to Nonstandard Analysis*, Academic Press, 1985
- [5] Iliadis, S.D., "Positions of continua on the plane and fixed points", *Vestnik Moscow, Univ. Ser.I Mat. Mekh.*, 1970, **25**, no. 4, 66-70
- [6] Lindstrom, T., "An invitation to nonstandard analysis", in *Nonstandard Analysis and Its Application*, ed. by N. Cutland, Cambridge University Press, 1988
- [7] Loeb, P. and Wolff, M., *Nonstandard Analysis for the Working Mathematician*, Kluwer, 2000
- [8] Mayer, J.C., "Embeddings and prime end structure of chainable continua", *Houston Journal of Mathematics*, 1982, **8** no. 2, 221-253
- [9] Sieklucki, K., "On a class of plane acyclic continua with the fixed point property", *Fund.Math.*, 1968, **63**, 257-278