A “Flatness” property for certain regions that contain no standard points

Steven C. Leth

University of Northern Colorado

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We will call an internal subset of the nonstandard plane \textit{IL-chainable} (for “infinitesimally linearly chainable”) if it can be covered by a hyperfinite collection of open sets $G_0, \ldots, G_n$ of infinitesimal diameter with the “linear chain” condition that $G_i \cap G_j$ is nonempty iff $|i - j| \leq 1$.

If $\delta > 0$ we will call a standard subset of the plane, or an internal subset of the nonstandard plane \textit{$\delta$-chainable} if it can be covered by a finite (or hyperfinite) collection of open sets $G_0, \ldots, G_n$ of diameter less than $\delta$ with the “linear chain” condition that $G_i \cap G_j$ is nonempty iff $|i - j| \leq 1$. 
If $\delta$ is not too small we might cover this with chained sets as follows:
If $\delta$ is a little smaller it might look like this:
But it is easy to see that we cannot use chaining sets with distance smaller than the length of the segment shown below:
By a **simple dense canal** in $E$ we mean an infinite arc $C$ contained in the complement of $E$ such that $C$ is the image of a one-to-one continuous function $c$ from $[0, \infty)$ to $\mathbb{R}^2$ that has the property that for all $\varepsilon > 0$ there exists $t_0$ such that for all $t \in [t_0, \infty)$, $c(t)$ is on a “transverse cross cut” of distance less than $\varepsilon$ (i.e. a segment whose endpoints are in $E$ on opposite sides of the arc and whose length is less than $\varepsilon$), and that is dense in $E$ (i.e. is such that $\overline{C} - C = E$). The notion of a simple dense canal is due to Sieklucki [5].

The next few slides illustrate a simple dense canal. The canal itself is in the complement of the set $E$, but the set $E$ consists of all the limiting points of the canal. The canal is actually an arc inside an open set in the complement that grows arbitrarily thin.
Simple dense canals are important in the Plane Fixed Point Problem which can be stated as follows. If $E$ is a non-separating plane continuum (i.e. a compact, connected subset of the plane whose complement is also connected) must it have the fixed point property? An extensive and comprehensive overview of results and questions related to the plane fixed point problem can be found in [2].

In 1967-1970 Bell, Sieklucki and Iliadis showed that if $f$ is any fixed point free mapping on such a set $E$ as above, then $E$ contains a simple dense canal in a subset $E'$ of the boundary of $E$ such that $f(E') = E'$ [1] [5] [3]. There is strong reason to believe that sets that arise naturally from such simple dense canals are IL-chainable. The value of this as a possible step toward the fixed point problem or open questions related to the problem is significant. We will come back to this point briefly at the end of this talk. This was the motivation for the following result:
Definition

We will say that a set \( A \subset \mathbb{R}^2 \) contains a **Y-set** if there exist four points \( a, b, c, \) and \( x \) in \( A \), arcs \( C_{ax}, C_{bx}, \) and \( C_{cx} \) in \( A \) intersecting only at \( x \), from \( a \) to \( x \), \( b \) to \( x \), and \( c \) to \( x \), respectively, such that none of the points \( a, b, \) or \( c \) are infinitesimally close to any point on the arcs joining the others to \( x \) (thus, for example, no point of \( C_{ax} \) is infinitesimally close to \( b \) or to \( c \)).

If \( \delta > 0 \) we will say that a set \( A \) in the plane contains a **size \( \delta \) Y-set** if there exist four points \( a, b, c, \) and \( x \) in \( A \), and arcs \( C_{ax}, C_{bx}, \) and \( C_{cx} \) in \( A \) intersecting only at \( x \) from \( a \) to \( x \), \( b \) to \( x \), and \( c \) to \( x \), respectively, such that none of the points \( a, b, \) or \( c \) are within \( \delta \) of any point on the arcs joining the others to \( x \) (thus, for example, no point of \( C_{ax} \) is within distance \( \delta \) of \( b \) or \( c \)).
If a set contains a size $\delta$ Y-set then it is easy to see that it cannot be $\delta$-chainable. The converse is not true. For each $n$ there exist sets that are not $n\delta$ chainable but contain no size $\delta$ Y set. Similarly if a set is IL-chainable then it contains no Y-set. However it is not known whether the converse is true or not.
If $\delta$ is smaller than this distance then the set is not $\delta$-chainable.

This set contains no $Y$-set of size greater than this distance.
Theorem

Let $V \subseteq \ast \mathbb{R}^2$ be a bounded (internal) region (i.e. contained in some $\ast B(0, r)$ for a standard real $r > 0$) bounded by a simple closed curve $S$ and suppose that $\overline{V}$ contains no standard points, $\text{st}(V)$ does not disconnect the plane, and that there exists an arc $A$ of infinitesimal length on $S$ such that $\ast \text{st}(V) \cap S \subseteq A$. Then $V$ contains no $Y$-set.
If we remove the condition that $V$ contains no standard points there are simple counterexamples. For example, we may let the blue line segments below form a standard set and the green border be $S$, where every point of $S$ stays within an infinitesimal of the blue set. The blue set itself is clearly a Y-set, and the other conditions of the theorem are satisfied.
If we move the blue set over and up by an infinitesimal amount, we cannot surround it by a border that intersects $^*st(V)$ only on one small arc.
Sketch of proof: Suppose that such a set $V$ contains a $Y$-set. We may choose our $a, b, c,$ and $x$ such that the small arc $A$ is not near $a, b,$ or $c$. We find balls $B_a, B_b, B_c$ satisfying the following conditions:

i) $a \in *B_a$, $b \in *B_b$, and $c \in *B_c$.

ii) Each of these three balls have non-infinitesimal radius.

iii) The arc $A$ does not intersect $\overline{B_a}, \overline{B_b},$ or $\overline{B_c}$

iv) every point on $\overline{B_a}$ is not near any point on $C_{bx}$ or $C_{cx}$, every point on $\overline{B_b}$ is not near any point on $C_{ax}$ or $C_{cx}$, and every point on $\overline{B_c}$ is not near any point on $C_{ax}$ or $C_{bx}$

v) The interior of each ball contains points not disconnected from infinity by $st(Y)$ together with the closure of the other two balls. For example, the interior of $B_a$ contains points not disconnected from infinity by $st(Y) \cup \overline{B_b} \cup \overline{B_c}$. 

![Diagram of the sketch of proof with labeled balls and arcs]
We let $Y'$ be the portion of the $Y$-set obtained by starting at $x$ and following each of the three paths to the first points of intersection with the three balls. We then define internal infinite arcs $\Gamma_a$, $\Gamma_b$ and $\Gamma_c$ that, along with $Y'$ divides the nonstandard plane into three regions as shown. We consider connected components of $\ast \mathbb{R}^2 - (S \cup \Gamma_a \cup \Gamma_b \cup \Gamma_c)$. One of these is $V$, which intersects all three of our newly constructed regions, but this is the only connected component of this set that can intersect more than one of these regions. The set $\ast \text{st}(Y')$ is connected, but there must be standard points of it in two different regions. We show that one of these standard points must be contained in a set whose boundary is completely contained in the complement of $\ast \text{st}(Y')$, contradicting the fact that it is connected.
We want to use this result to show that we can “close off” portions of a simple dense canal with uncomplicated arcs to obtain sets that contain no Y-set. In the theorem below we use arcs of circles to close off the regions.

**Theorem**

Let $E$ be a non-separating plane continuum that contains a simple dense canal $C$. Then there exist points $p_1 \approx p_2$ on $C$ and an arc $A$ of a circle such that the portion of $C[p_1, p_2]$ together with $A[p_2, p_1]$ forms a simple closed curve $S$ that is infinitesimally close to every point in $E$ and is such that $V_S$ contains no standard points in $E$. 
We go back to our example of a simple dense canal. We want to close off a portion with an arc of a circle in such a way that the region contains no standard points (and thus, by the previous theorem, contains no Y-set).
By saturation, we really need to show the following standard result:

For any finite collection of standard points \( \{x_1, x_2, \ldots, x_n\} \) in \( E \) and any standard \( \delta > 0 \) there exist points \( a \) and \( b \) on \( C \) and an arc of some circle \( A \) such that \( C[a, b] \) is within \( \delta \) of every point of \( E \), \( a \) and \( b \) are within \( \delta \) of each other, and the portion of \( C \) from \( a \) to \( b \) together with \( A[b, a] \) forms a simple closed curve whose interior does not contain any of the points \( \{x_1, x_2, \ldots, x_n\} \).
Nonstandard methods still seem to be very helpful in proving this standard result. The picture below illustrates why. Suppose that the ball shown is standard, but that the part of $\ast C$ shown is within $\delta$ of every point of $E$. 
Then this must repeat infinitely often with later portions of the curve. This allows us to create (many annoying details later) infinitely many disjoint regions that all are within $\delta$ of every point of $E$. 
The two theorems together give us the following standard result:

Theorem

Let $E$ be a non-separating plane continuum that contains a simple dense canal, and let $C$ be an infinite arc (i.e. the image of a one-to-one continuous function from $[0, \infty)$ to $\mathbb{R}^2$) in the complement of $E$ with the property that $\overline{C} - C = E$ and for every $\varepsilon > 0$ there exists a point $p$ on $C$ such that all points on the arc beyond $p$ are on a transverse cross cut of distance less than $\varepsilon$. Then for any $\delta > 0$ there exist points $p_1$ and $p_2$ on $C$, and an arc of a circle $A$, such that $C[p_1, p_2]$ together with $A[p_1, p_2]$ forms a simple closed curve $S$ that is within $\delta$ of every point in $E$ and is such that $V_S$ contains no size $\delta$ $Y$-set.
The conjecture is that all of the statements involving no Y-sets (or no size \( \delta \) Y sets) can be improved to IL-chainability (\( \delta \) chainability). Thus, the immediate next goal is to prove this conjecture:

Let \( V \subset *\mathbb{R}^2 \) be a bounded region bounded by a simple closed curve (internally) and suppose that \( \overline{V} \) contains no standard points, \( st(V) \) does not disconnect the plane, and that there exists an arc \( A \) of infinitesimal length on \( \partial(V) \) such that \( *st(V) \cap \partial(V) \subset A \). Then \( V \) is IL-chainable.
It seems quite likely that the theorem can be improved to the statement above even if being IL-chainable does not follow from the condition “contains no Y-set.” This is because the condition that $V$ contain no standard points and has small intersection with $^*\text{st}(V)$ is quite strong and precludes, for example, the pursuit of any counterexample in which there are long patches of many thin long loops that would create a standard part with an interior. For example, if we have a part like in an earlier example, but the distances are infinitesimal while the $x$-direction is noninfinitesimal there will be large intersection of the boundary of $V$ and $^*\text{st}(V)$. 
The conjecture is important in the pursuit of the plane fixed point problem and its sub-questions because chained sets form “traps” for fixed points. For example, in the picture below, if $p$ and $f(p)$ are both on the inside portion of the simple dense canal, and this is a portion that is IL-chainable, then $f$ must have a fixed point. Note that all these closed off portions contain points of $E$ on the inside portion infinitesimally close to every point of the canal.


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