sumsets contained in sets of positive density

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In this talk I will outline some results obtained by Di Nasso, Goldbring, Jin, Leth, Lupini and Mahlburg. This research was supported by an “AIM SQuaRE” grant from the American Institute of Mathematics. The paper will appear in the Canadian Journal of Mathematics.
Nonstandard analysis, first developed by Abraham Robinson in the early 1960’s, makes use of the fact that there are other models besides the usual ones that satisfy all the same mathematical statements that can be made in First Order Logic. The goal is to exploit what is different in these models - the existence of a wide array of “actualized” limits including “infinite” numbers - and also what is the same - all mathematical properties that can be expressed formally in first order logic. Sometimes we might make arguments that are “internal,” for example taking advantage of the fact that sums of infinitely many terms still act like finite sums. Other times we might use arguments that take advantage of an “external” view of our model - for example by using measures or equivalence relations that are not recognized entities inside the model.
To use the “best of both worlds” approach to maximum effect we usually work in nonstandard models using a large language that includes symbols for every standard object we are interested in. Every subset of the natural numbers is named, for example. This helps us make strong use of the **transfer principle**: the nonstandard model is elementarily equivalent to the standard model, so every (first order) mathematical statement expressible in our large language is true in the nonstandard model iff it is true in the standard model. We also usually work in sets that have some level of saturation, i.e. that have the property that collections of internal sets with the finite intersection property have a nonempty intersection, assuming the cardinality of the collection is not too large.
The nonstandard counterpart of a standard set $E$ is denoted by $^*E$. An **internal** set is one that can be defined inside the nonstandard model using standard sets and functions and other internal parameters. Equivalently, it is a subset of $^*P(\mathbb{N})$, where $P(\mathbb{N})$ is the power set of $\mathbb{N}$. Elements of $P(\mathbb{N})$ that are not in $^*P(\mathbb{N})$ are **external** sets. The set of “infinite” natural numbers (i.e. the set $^*\mathbb{N}\setminus\mathbb{N}$) is an example of an external set. By the transfer principle every internal nonempty set has a least element since that statement is true of subsets of the natural numbers. Of course the nonstandard model “thinks” that all elements are finite and has no definable way to tell the difference between standard and “infinite” elements.
As a simple example of the use of nonstandard methods in combinatorics we look at a proof of Ramsey’s theorem due to Hirshfeld [1988]. For simplicity we consider the case of two dimensions and three colors. So, suppose all pairs of natural numbers are colored either red, green or blue. We choose any nonstandard model of the natural numbers and an \( H \in \mathbb{N} \). Every standard natural number \( m \) can be assigned a color based on the coloring of the pair \( \{m, H\} \) in the nonstandard model. This partitions \( \mathbb{N} \) into a union of disjoint sets \( R, G, \) and \( B \). The statement

\[
\mathbb{N} = R \cup G \cup B
\]

is a first order mathematical statement, so by transfer:
We must have that

\[ \mathbb{N}^* = R^* \cup G^* \cup B^*, \]

so that \( H \) is in one of the sets, and wlog we will say that \( H \in B^* \).
We may now choose any \( b_1 \in B \), and inductively define a monochrome set
as follows:

Given \( \{b_1, b_2, \ldots b_n\} \), \( H \) is witness to the fact that there exists some larger
element of \( B^* \) that is colored blue with each of \( b_1, b_2, \ldots b_n \). By transfer
there is some larger element of \( B \) with this property, and we may choose
any of them to be \( b_{n+1} \).

It is not surprising that this proof is very reminiscent of the ultrafilter
proof, as ultrafilters are often at the heart of nonstandard constructions,
and nonstandard elements can be viewed as ultrafilters. Mauro Di Nasso
will say more about this in his talk.
The proof above uses only the transfer principle. Most applications use additional tools, and probably the most important one is the concept of Loeb measure. We first note that every finite nonstandard real number $r$ is within an infinitesimal of exactly one real number, and this real number is denoted by $\text{st}(r)$.

Let $H \in *\mathbb{N} \setminus \mathbb{N}$. There is a natural internal counting measure on any interval of the form $[a + 1, a + H]$, obtained by dividing the internal cardinality of an internal set by $H$ and taking the standard part, i.e. for every internal set $E$ contained in $[a + 1, a + H]$, the measure of $E$ is defined to be $\mu(E) := \text{st}(\frac{|E|}{H})$, where $\text{st}$ is the standard part mapping. This defines a finitely additive measure on the algebra of internal subsets of $[a + 1, a + H]$, which canonically extends to a countably additive probability measure on the $\sigma$-algebra of Loeb measurable subsets of $[a + 1, a + H]$, and we will also write $\mu$ for this extension (or $\mu_I$ if we want to emphasize the interval $I$ of interest).
Let $A$ be a standard subset of $\mathbb{N}$. The lower (asymptotic) density of $A$ is:

$$d(A) := \liminf_{n \to \infty} \frac{|A \cap [1, n]|}{n};$$
the upper (asymptotic) density of $A$ is:

$$
\overline{d}(A) := \limsup_{n \to \infty} \frac{|A \cap [1, n]|}{n};
$$
the (upper) Banach density of $A$ is:

$$BD(A) := \lim_{n \to \infty} \sup_{x \in \mathbb{N}} \frac{|A \cap (x + [1, n])|}{n}.$$
Nonstandard equivalents of these standard notions:

**Proposition**

1. \( \overline{d}(A) \geq \alpha \) iff there exists an \( H \in \mathbb{N} \setminus \mathbb{N} \) such that \( \mu_{[1,H]}(*A) \geq \alpha \).

2. \( d(A) \geq \alpha \) iff for all \( H \in \mathbb{N} \setminus \mathbb{N} \) \( \mu_{[1,H]}(*A) \geq \alpha \);

3. \( BD(A) \geq \alpha \) iff there exists \( H \in \mathbb{N} \setminus \mathbb{N} \) and \( x \in \mathbb{N} \) such that \( \mu_{x+[1,H]}(*A) \geq \alpha \).
Proof of number 2, as an example:
\[ \mathcal{d}(A) \geq \alpha \text{ iff for any } \epsilon > 0 \text{ there exists } n_\epsilon \in \mathbb{N} \text{ such that for all } n > n_\epsilon \]

\[ \frac{|A \cap [1, n]|}{n} \geq \alpha - \epsilon. \]

By transfer, this is true iff for all \( H \in \ast \mathbb{N} \setminus \mathbb{N} \) and every standard \( \epsilon > 0 \)

\[ \frac{|\ast A \cap [1, H]|}{H} \geq \alpha - \epsilon, \]

which is equivalent to \( \mu_{[1,H]}(\ast A) \geq \alpha. \)
Hindman’s theorem tells us that if the natural numbers are partitioned into finitely many sets, then at least one of these sets contains an IP-set, i.e. contains all finite (non-repeating) sums of some infinite set. Erdős was interested in density versions of this result. A counterexample of E. Straus, as reported in [Erdős 1980], shows that there are sets of positive lower density that do not contain a translate of an IP set. Along these lines Erdős asked the following question (see, for example, [Nathanson 1980] or [Erdős and Graham 1980]):

If $A \subseteq \mathbb{N}$ is of positive lower density must there exist infinite sets $B$ and $C$ such that $B + C \subseteq A$?

The following theorem provides a partial answer to this question.
Theorem

[DGJLLM] Suppose that $A \subseteq \mathbb{N}$ is such that $BD(A) > \frac{1}{2}$. Then there are infinite $B, C \subseteq \mathbb{N}$ with $C \subseteq A$ such that $B + C \subseteq A$.

Using this result along with a Ramsey theory (or ultrafilter) argument we obtain:

Theorem

[DGJLLM] If $A \subseteq \mathbb{N}$ is such that $BD(A) > 0$, then there exist infinite sets $B, C \subseteq \mathbb{N}$ and $k \in \mathbb{Z}$ such that $B + C \subseteq A \cup (A + k)$.

Note that the condition “$BD(A) > 0$” is significantly weaker than the assumption that $A$ has positive lower density. These results can be extended to arbitrary countable amenable groups. In particular they are true in every $\mathbb{Z}^n$.
The proof uses the following result of Bergelson [1985]:

**Theorem**

[Bergelson] Suppose that \((X, \mathcal{B}, \mu)\) is a probability space and \((A_n)\) is a sequence of measurable sets for which there is an \(a > 0\) such that \(\mu(A_n) \geq a\) for each \(n\). Then there is an infinite \(P \subseteq \mathbb{N}\) such that, for every finite \(F \subseteq P\), we have \(\mu(\bigcap_{n \in F} A_n) > 0\).

We will apply this result using Loeb measure on a hyperfinite interval, but first we need a few observations and lemmas.
Jin [2001] applied the Birkhoff ergodic theorem to show that if $A$ has Banach density $\alpha > 0$ then there is an interval $I$ of $^{*}\mathbb{N}$ of hyperfinite length such that for $\mu_I$ almost all $x \in I$, the standard portion of the set $^{*}A - x$ has lower density $\alpha$. For this result we first show that there exists an interval $I$ of $^{*}\mathbb{N}$ and an $x \in I$ such that if we let $L$ denote the standard portion of the set $^{*}A - x$, then both

$$\mu_I({^*}A) = \alpha \text{ and } \mu_I({^*}L) = \alpha.$$ 

If we write $L = (l_i)$, it is easy to see that there exists a subset $D = (d_i)$ of $A$ such that $l_i + d_n \in A$ for $i \leq n$. To see this note that for any finite subset $F$ of $\mathbb{N}$, $x$ is a witness to the fact that there exist arbitrarily large elements $a$ in $A$ such that $l_i + a \in A$ for each $i \in F$. 
Now for each \( d_n \in D \) we have:

\[
\mu(I \cap (A - d_n) \cap I_v) \geq 2\alpha - 1 > 0.
\]

Letting \( a = 2\alpha - 1 \) and applying Bergelson’s result we may, after passing to a subsequence of \((d_n)\), assume that, for every \( n \in \mathbb{N} \), we have

\[
\mu\left(\bigcap_{i \leq n} (A - d_i) \cap I_v\right) > 0.
\]

In particular, this implies that, for every \( n \in \mathbb{N} \), we have \( L \cap \bigcap_{i \leq n} (A - d_i) \) is infinite.
We may now take $b_1 \in L$ arbitrary and take $c_1 \in D$ such that $b_1 + c_1 \in A$. Fix $b_2 \in (L \cap (A - c_1)) \setminus \{b_1\}$ and take $c_2 \in D$ such that $\{b_1 + c_2, b_2 + c_2\} \subseteq A$. Take $b_3 \in (L \cap (A - c_1) \cap (A - c_2)) \setminus \{b_1, b_2\}$ and take $c_3 \in D$ such that $\{b_1 + c_3, b_2 + c_3, b_3 + c_3\} \subseteq A$. Continue in this way to construct the desired $B$ and $C$. 
The shift result for small Banach density (short version): If $A$ has positive Banach density then we can look at the density of blocks of length $n$ that intersect $A$, and for large enough $n$ the Banach density of this set (relative to the set of all blocks of that length) will be greater than $1/2$. So, by the previous theorem there exist sets $B$ and $C$ and a finite number $n$ such that $B + C \subset A + [0, n - 1]$. Using $n^2$ colors we may code every pair of natural numbers based on the distance from $b_i + c_j$ to $A$ and from $b_j + c_i$ to $A$. By Ramsey’s theorem we may pick an infinite monochromatic subset $M$ of $\mathbb{N}$. If we let $B' = \{ b_i \in B : i \in M \text{ and } i \text{ is odd} \}$ and $C' = \{ c_i \in C : i \in M \text{ and } i \text{ is even} \}$ then $B' + C'$ is contained in the union of at most two shifts of $A$, and we may translate $B'$ to obtain the desired result.


H. Furstenberg, *Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions,* J. Analyse Math. 31 (1977), 204-256.


