From: *Young mathematicians at work: Constructing number sense, addition, and subtraction* by C. T. Fosnot, & M. Dolk, pages 51-54.

4 PLACE VALUE ON THE HORIZON

One is hard-pressed to think of universal customs that man has successfully established on earth There is one, however, of which he can boast—the universal adoption of the Hindu-Arabic numerals to record numbers In this we perhaps have man's unique worldwide victory of an idea

—*Howard W. Eves*, Mathematical Circles Squared

Numbers have neither substance, nor meaning, nor qualities They are nothing but marks, and all that is in them we have put into them by the simple rule of straight succession

—*Hermann Weyl quoted in* Armchair Science Reader

ADDITIVE NUMERATION SYSTEMS

How do we comprehend and communicate "how many?"? Although the eye can often perceive five or fewer objects as a whole (see Chapter 3), amounts larger than this need to be counted, or the larger amount needs to be decomposed into smaller amounts that can be subitized and then added Because we cannot "see" quantities larger than four or five as a whole but instead must operate on them to determine how many, humans across cultures and over time constructed ways to represent amounts symbolically We invented numerals and operations

Sticks, Stones, and Bones

The first numerical marks that we are aware of in history come from the Paleolithic era These marks were slashes, or tallies, carved onto cave walls or into bone, wood, or stone One slash meant one object; thus ten reindeer were denoted with ten tallies The marks had a one-to-one correspondence with the objects being counted Bones bearing such numerical slashes have been found that are nearly 30,000 years old (Guedj 1996)

Because a "one thing, one notch" system was too cumbersome to represent large amounts, over time humans refined this system One refinement was the use of knots arranged along cords, in Persia, in the fifth century B C

(Guedj 1996) By the thirteenth century the Incas had refined this system, developing a *quipu*—a cord held horizontally from which knotted strings hung The type of knots used, the length of the cord, and the color and position of the strings all communicated levels of quantities: single units, tens, and hundreds Some cultures used different-shaped stones to represent different amounts, while others made objects out of clay Sumerian clay stones known as *calculi* (*calculus* is Latin for stone) have been found that date to the fourth millennium B C A small clay cone was used to represent a value of one; a ball, ten; and a large cone, sixty When a contract specifying a particular amount was agreed upon, the calculi representing the sum of that number were placed inside a hollow ball Notches representing the calculi inside were made on the surface of the seal (Guedj 1996)

Almost all these early number systems used ones, fives, tens, and twenties, but when one realizes that early counting was usually done with fingers and toes, this isn't so surprising

The Invention of Numerals

The first written mathematical symbols of which we are aware appeared in early Babylonian times (around 3300 B C) A nail shape represented units, a chevron shape represented tens Nine nails and one chevron thus represented nine and ten, the quantity 19 Over time and across cultures, similar written systems were developed Although new symbols were invented to show quantity groupings, rather than single units, the number of shapes drawn was still in a one-to-one correspondence, either to units or number of groups And different symbols were used for different-sized groups For example, the Mayans used a bar to equal five and a dot to represent units They wrote 19 with three bars and four dots Ancient Egyptians used lines to represent ones, a basket handle to represent tens, a coiled rope to represent hundreds, and a lotus flower blooming on its stalk to represent thousands They wrote 19 with one basket handle and nine lines All of these systems are examples of additive numeration The operation of addition is employed; the value of the number is equal to the sum of the values of the symbols Each symbol is repeated the number of times it must be added (Guedj 1996)

Roman numerals are also considered an additive numeration system, in that the symbols represent the worth of the group of objects and the total amount is the sum of the symbols C represents one hundred; L, fifty; X, ten; V, five; and I, one, and these symbols are repeated the number of times they must be added One advance exists in this system, though: placing smaller-quantity symbols *before* larger-quantity ones denotes subtraction That is, XI denotes addition ($10 + 1 = 11$), but IX denotes subtraction ($10 - 1 = 9$) This saved a little of the tedium and cumbersome writing when many symbols were needed to represent large amounts Nevertheless, even with these advances in the writing of numbers (which took thousands of years), the simplest calculation remained arduous

The Invention of Place Value

In early numeration systems, the value of the digit had little or no relation to the position in which it was placed. Even in the roman numeral system, although placement could denote subtraction, I still meant one, whether it was placed before or after the X. C always equaled one hundred no matter where it was placed: MCI meant one thousand one hundred and one; MCCCI meant one thousand three hundred and one. The amounts designated by the symbols were simply combined.

The positional notation that characterizes our number system today was a big idea in the evolution of number systems. The idea employs the operation of multiplication. For example, the digit 2 in the second column to the left stands for two tens, but when placed in the third column to the left, it stands for two hundreds. No separate symbols are needed to represent tens or hundreds.

The numerals 1 through 9 appeared in India in inscriptions from the third century B.C., but the symbol for none, and the idea of zero, had yet to be invented. The combination of positional notation and the idea of zero in India in the fifth century A.D., which passed via the Arabs to Europe, produced a powerful new system of notation, one that led to advances in calculating and to the development of modern mathematics. In the ninth century the Arab mathematician Muhammad ibn Musa al-Khwarismi wrote a book, *The Book of Addition and Subtraction by Indian Methods*, presenting these new ideas. The book became extremely famous in Europe and was eventually translated into Latin in the twelfth century, thus establishing column arithmetic, using borrowing and carrying, as the method of calculating. Over time column arithmetic became known as *algorism*—the Latin name for al-Khwarismi (Guedj 1996). Today we use the term *algorithm*.

Why Did the Development of Place Value Take So Long?

What makes place value so difficult? Why did it take so long to be developed? For one thing, the idea of zero is conceptually different from all previously developed numbers in that it is not connected to real objects. Piaget noted that the concept of zero introduced number as an idea in itself, separate or abstracted from concrete reality (cited in Guedj 1996). Then, too, the idea of zero evolved in stages. First it was simply functional, a symbol that represented what happened to a number when it was multiplied by ten (324 × 10 became 3240). Later it was used to stand for the absence of objects in column notation. Only much later in its development did it become a number of its own, defined mathematically as $n - n$ (Guedj 1996).

Children have this same struggle with zero. Recall how Madeline's children figured out that a necklace with 20 beads could be bought with two dimes and no pennies, yet they still thought of 30 and 40 as whole quantities, not as three dimes and no pennies, or four dimes and no pennies. They needed to check out Ellie's conjecture about the zero with other numbers. They did not automatically see that the zero meant no pennies. Children also often write numbers above 100 with two zeroes—using 10013 for one hundred and thirteen, for example. They do not fully understand the combination of the place value columns and the use of zero.

In addition, the idea that a numeral can represent ones or tens or hundreds, depending on where it is placed, involves the big idea of unitizing. The numeral 2 represents two units, but the units themselves can change; they can be ones or tens or hundreds or thousands or more. Cognitively, this is another abstraction from concrete objects. The unit is a variable. Its amount changes depending on the column in which it is placed. The numeral 2 simply represents the cardinality of the units.

HELPING CHILDREN DEVELOP MATHEMATICAL NOTATION

Just as these ideas were difficult for humans to invent, evolving only slowly over many, many years, they are huge developmental milestones in the mathematical development of young children. Martin Hughes (1986) showed children between the ages of three and seven several different cans containing different-sized groups of plastic bricks (one, two, three, five, and six) and asked them to put something on paper to show how many bricks were in each of the cans. The developmental progression he found paralleled the historical progression of the development of numerical writing.

Many of the youngest children made idiosyncratic drawings that seemed to have no connection at all to quantity. They just drew pictures of the objects with no attempt to represent the amount. As the big idea of one-to-one correspondence was constructed, children began to represent quantity with pictographic representations—they actually drew the bricks, one for one, to show the amount. Later the representations took on an iconic representation, with slash marks or dots used as *symbols* to represent the quantity of the objects. Eventually, but not without a great deal of struggle, children attempted to represent the quantity with only one symbol. Understanding that one symbol can represent the whole amount requires an understanding of cardinality. This is a landmark leap for children.